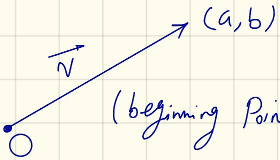
"Vectors in the Plane and Space"

Vector: magnitude & Direction

On the Plane, each vector can be represented by two Coordinates

$\vec{v} = (a, b) \qquad a, b \in \mathbb{R}$



(beginning Point is not important)

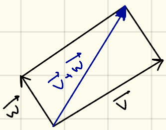
In space, we need 3 coordinates

$\vec{v} = (a, b, c)$

Addition:

Add two vectors coordinate wise

$(a,b,c) + (d,e,f) = (a+d,\ b+e,\ c+f)$



$\Rightarrow \vec{v} + \vec{w} = \vec{w} + \vec{v}$ (Addition is Commutative)

Addition is also associative

$\vec{v} + (\vec{w} + \vec{u}) = (\vec{v} + \vec{w}) + \vec{u}$

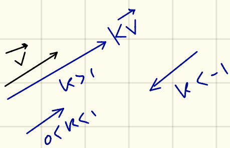
$\vec{0}$: $\vec{0} + \vec{v} = \vec{v}$ for all $\vec{v}$

↳ Zero vector

$\vec{0}$ is the identity element for vector addition.

Multiplication by a Scalar:

Say $k \in \mathbb{R} \Rightarrow k(a,b,c) = (ka, kb, kc)$



Some Properties

$k(\vec{v}+\vec{w}) = k\vec{v}+k\vec{w}$

$(k+l)\vec{v} = k\vec{v}+l\vec{v}$

$1.\vec{v} = \vec{v}$

Abstract Definition

A Set $\mathcal{V}$ (whose elements will be called vectors) is a vector space.
if:

- There are two operations
  - $+$ addition of vectors
  - $\cdot$ multiplication of a vector by scalar.

That satisfy the following list of axioms:

$$\left(\begin{array}{l} + \text{ means for } v,w \in \mathcal{V} \text{ we define } v+w \in \mathcal{V} \\ \cdot \text{ means for } v \in \mathcal{V},\ c \in \mathbb{R},\ c.v \in \mathcal{V} \end{array}\right)$$

Axiom 1:

$v+w = w+v$ for all $v, w \in V$

Axiom 2.

$v+(w+u) = (v+w)+u$ for all $v, u, w \in \mathcal{V}$

Axiom 3. There is a vector denoted $\vec{0}$ such that
$\vec{0}+\vec{v} = \vec{v}$ for all $v \in \mathcal{V}$

Axiom 4. For every $v \in \mathcal{V}$, there is $-v \in \mathcal{V}$ such that

$$v+(-v) = \vec{0}$$

Axiom 5.

If $c \in \mathbb{R}$, $v, w \in \mathcal{V}$ then $c.(v+w) = (c.v)+(c.w)$

Axiom 6.

If $c_1, c_2 \in \mathbb{R}$, $v \in \mathcal{V}$ then

$$(c_1 + c_2) \cdot v = (c_1 \cdot v) + (c_2 \cdot v)$$

Axiom 7.

If $c_1, c_2 \in \mathbb{R}$, $v \in \mathcal{V}$ then
$(c_1 c_2) \cdot v = c_1 \cdot (c_2 \cdot v)$

Axiom 8.

$1 \cdot v = v$ for all $v \in \mathcal{V}$

Some Basic Properties

Lemma: The zero vector in axiom 3 is unique (there is only one vector that satisfies the condition in axiom 3)

Proof. Suppose we have $\vec{0}_1$, $\vec{0}_2$ both satisfying the condition in axiom 3

$\vec{0}_1 + \vec{0}_2 = \vec{0}_2$ (b/c $\vec{0}_1$ is the identity element)
$\vec{0}_1 + \vec{0}_2 = \vec{0}_2 + \vec{0}_1$ by Ax. 1
$= \vec{0}_1$ (b/c $\vec{0}_2$ is the identity element.)
$\Rightarrow \vec{0}_1 = \vec{0}_2$

Lemma: for every $v \in \mathcal{V}$, there is a unique $-v \in \mathcal{V}$ satisfying the condition in Axiom 4.

Proof.
Suppose, for a given $v \in \mathcal{V}$, we have $(-v)_1$ and $(-v)_2$ satisfying the condition in Axiom 4.

$$\overbrace{(-v)_1 + \overbrace{(v + (-v)_2)}^{\vec{0}}}^{(-v)_1 \text{ by A3 \& A1}} \overset{A2}{=} \overbrace{\overbrace{((-v)_1 + v)}^{\vec{0}} + (-v)_2}^{(-v)_2 \text{ by A3 \& A1}} \Rightarrow (-v)_1 = (-v)_2$$

Lemma: $0 \cdot v = \vec{0}$ for all $v \in V$

Proof.

$$0 \cdot v = (0+0) \cdot v$$
$$= (0 \cdot v) + (0 \cdot v) \quad A6$$

Add $-(0 \cdot v)$ to both sides by A4

$$-(0 \cdot v) + (0 \cdot v) = -(0 \cdot v) + ((0 \cdot v) + (0 \cdot v))$$
$$\vec{0} = (-(0 \cdot v) + (0 \cdot v)) + (0 \cdot v)$$
$$\vec{0} = \vec{0} + (0 \cdot v)$$
$$\Rightarrow \vec{0} = 0 \cdot v$$

Cartesian K-Dimensional Space.

$V = \mathbb{R}^k$, k is a positive integer

$\mathbb{R}^k = \{(a_1, a_2, a_3, \ldots, a_k)$ where $a_1, \ldots, a_k \in \mathbb{R}\}$ set to K-toples

Addition: $(a_1, a_2, \ldots, a_k) + (b_1, b_2, \ldots, b_k)$
$= (a_1 + b_1, a_2 + b_2, \ldots, a_k + b_k)$

Multiplication.

$$c \cdot (a_1, \ldots, a_k) = (ca_1, ca_2, \cdots, ca_k)$$

Proof that $\mathbb{R}^k$ is a vector space.

One by One, show all the axioms A1-A8 are satisfied.

A1. For each $v = (a_1, \cdots, a_k)$
$w = (b_1, \ldots, b_k)$

$$\left.\begin{array}{l} v+w = (a_1+b_1, \ldots, a_k+b_k) \\ w+v = (b_1+a_1, \ldots, b_k+a_k) \end{array}\right\} \text{Same} \Rightarrow v+w = w+v$$
$$(a_1+b_1, \ldots, a_k+b_k) = (b_1+a_1 + \cdots + b_k a_k)$$

A3. Set $\vec{0} = (0, 0, \ldots, 0)$

$\vec{v} + \vec{0} = (a_1+0, \cdots a_k+0)$

$= (a_1, a_2, \cdots, a_k) = v$

through all axioms $\mathbb{R}^k$ is a vector space.

Ex. $V = \{(a,b) \text{ where } a,b \in \mathbb{R}\}$

Addition : $(a,b) + (c,d) = (a+c, b+d)$

Scalar Multi : $r.(a,b) = (ra, 0)$

Is V with these Operations a Vector Space?

check A8 ($1A = A$) ?

$$1.(a,b) = (1.a, 0) \neq (a,b)$$

if $b \neq 0$ $\Rightarrow$ V is not a vector Space.

Ex. V = Set of Positive Numbers

Addition : $a \oplus b = ab$

Scalar Multiplication: $r \odot a = a^r$

Is V a vector Space?

A1. $a \oplus b = ab$

$b \oplus a = ba$ $\Rightarrow ab = ba \Rightarrow a \oplus b = b \oplus a$

A2. $(a \oplus b) \oplus c = ab \oplus c = abc$ ✓

$a \oplus (b \oplus c) = a \oplus bc = abc$

A3. $a \oplus 0 = a.1 = a$

$\bar{0} = 1$

A4. $a \oplus -a = \bar{0} = 1 \Rightarrow a.\frac{1}{a} = 1$

$-a = \frac{1}{a} \in$ a Positive real number

A5. $r.(a \oplus b) = (ab)^r = a^r . b^r = (r \odot a) \oplus (r \odot b)$

A6. $(r+s) \odot a = rs \odot a = a^{r+s} = a^r . a^s = (r \odot a) \oplus (s \odot a)$

A7. $(rs) \odot A \stackrel{?}{=} r \odot (sA) =$

$A^{rs} = (A^s)^r = (sa)^r = r(sa)$

A8. $1 \odot a = a^1 = a$ ✓

" Remark "

In any vector space V, the generalized Commutative associative rules hold

$$((v_1+v_2)+(v_3+v_4)) = V_1+((V_2+V_3)+V_4)$$

etc.

We Can rearrange Paranthesis as we like.

Generalized Commutative rule:

Places of vectors in a Sum Can be permuted in anyway.

$$V_1+V_2+V_3+V_4 = V_4+V_3+V_2+V_1 \quad \text{etc.}$$

A Few things about Sets:

Let S, T be two sets

. $S \cap T$: S intersection T)

$x \in S\cap T \iff x \in S$ and $x \in T$

. $S \cup T$: Union of S and T

$x \in S \cup T \iff x \in S$ or $x \in T$

. $S \subset T$: S is a subset of T

$x \in S \Rightarrow x \in T$

Ex. Prove that $S \cap (T \cup R) = (S\cap T) \cup (S \cap R)$

1) $S \cap (T \cup R) \subset (S \cap T) \cup (S \cap R)$

$x \in S \cap (T \cup R) \Rightarrow x \in S$ and $x \in (T \cup R)$

$\Rightarrow x \in S$ and $(x \in T$ or $x \in R)$

$\Rightarrow x \in S$ and $x \in T$ or $x \in S$ and $x \in R$

$\Rightarrow x \in S\cap T$ or $x \in S \cap R$

$\Rightarrow x \in (S\cap T) \cup (S \cap R)$

2) $(S\cap T) \cup (S \cap R) \subset S \cap (T \cup R)$

Ex. Polynomial of degree $\leq n$

$$P_n(\mathbb{R}) = \{a_0 + a_1 x + \cdots + a_n x^n\}, \quad a_1, a_2, \ldots, a_n \in \mathbb{R} \quad (\text{think of } x \text{ as a symbol})$$

Addition: $(a_0 + a_1 x + \cdots + a_n x^n) + (b_0 + b_1 x_1 + \cdots + b_n x^n)$

$$= (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n) x^n$$

Scallar Multiplication: $r_1 (a_0 + a_1 x + \cdots + a_n x^n)$

$$= r a_0 + (r a_1) x + \cdots + (r a_n) x^n$$

also A1 to A8 should be satisfied.

Ex. Function on S.

S is a set $Fun(S) (= Fun(S \to \mathbb{R}))$

is the set of all function on S with real values.

Addition: $(f+g)(s) = f(s) + g(s)$

($f+g$ is the function whose value at s is $f(s) + g(s)$)

Scalar Mult: $(r \cdot f)(s) = r(f(s))$

Ex. Take $S = \{(0,0), (0,1), (1,0), (1,1)\}$

(1,0)	(1,1)
(0,0)	(0,1)

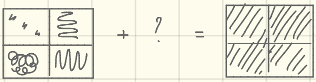
$f \in Fun(S)$ is determined by 4 values.

$f(0,0), f(0,1), f(1,0), f(1,1)$

If g is another vector $\Rightarrow (f+g)(1,0) = f(1,0) + g(1,0)$

Ex. Say (i,j) is "black" when $f(i,j)$ is 0

Say we have a picture



$f \quad - \quad f \quad = \vec{0}$

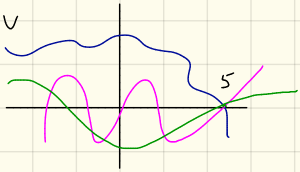
Ex. Let V be the following subset of Fun $(\mathbb{R})$

$V = \{ f \mid f(5) = 0 \}$

Such that

Addition: of Fun $(\mathbb{R})$

Scalar Multi: of Fun $(\mathbb{R})$



If V a vector space?

If we add two elements of V, is it back in V?

Say $f, g \in V$
$f(5) = 0, g(5) = 0$ $\Big\} \Rightarrow (f+g)(5) = f(5) + g(5) = 0 + 0 = 0$ ✓

chapter 2. Exercise 12.

Proof if a vector space has two elements then it has infinite many?

(why not with one)

if $\bar{v}, \bar{w}$ are two diffrent vectors in V.S

Then one of them is not Zero.

without loss of generality $v \neq 0$

Claim: For every $k \neq 0, 1$ $k \cdot v \neq 0$ or v

(Follow from uniqueness)

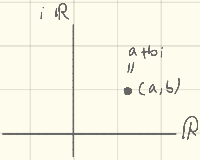
Chapter 3. Examples of V.S

Ex. $\mathbb{C} = \{a+bi\}$ as a vector space

Vector $\longrightarrow$ Complex #

Vector Sum $\longrightarrow$ Addition (component wise)

vector Scaling $\longrightarrow$ Mult. by real # (component wise)



$$\begin{cases} 0 = 0 + 0i \\ -(a+bi) = -a-bi \end{cases}$$

Ex. $Fun(S) = \{ f: S \to \mathbb{C} \}$
(complex vector function)

vector $\longleftrightarrow$ function

vector sum $\longleftrightarrow$ addition in $\mathbb{C}$ of functions

vector scaling $\longleftrightarrow$ multiplication by real #

$\underline{0}$ = the function with $\underline{0}(s) = 0+0i$ for all s

$\underline{-f}$ = the function with $f(-s) = -f(s)$ $\forall s$.

Exercise 10. If V is V.S, then so is $Fun_V(S) = \{f: S \to V\}$

Notation: $\oplus_V$ for sum in V

$\oplus_{Fun}$ for sum in function.

A1. $\left.\begin{array}{l} (f \oplus_{Fun} g)(s) = f(s) \oplus_V g(s) \\ (g \oplus_{Fun} f)(s) = g(s) \oplus_V f(s) \end{array}\right\} \Rightarrow \oplus_V$ is commutative

Proof other axioms.

Ex 3. $P(\mathbb{R}) = \{$Polynomial with $\mathbb{R}$-coefficient$\}$

vector $\longleftrightarrow$ Polynomial

vector sum $\longleftrightarrow$ polynomial +

vector scalar $\longleftrightarrow$ multiply by vector

NOTE: $P_k(\mathbb{R}) \subseteq P(\mathbb{R}) \subseteq Fun(\mathbb{R})$

Ex 4. $\text{Fun}(S,T) = \left\{ \begin{array}{l} f: S \longrightarrow R \\ f(t) = 0 \text{ for } t \in T \end{array} \right\}$

NOTE: $\text{Fun}(S,T) \subseteq \text{Fun}(S)$
(checking axioms for $\text{Fun}(S,T)$
looks same as for $\text{Fun}(S)$)

Exercise 8. Proof: $\text{Fun}(S,A) \cap \text{Fun}(S,B) = \text{Fun}(S, A \cup B)$

Proof:
$(\subseteq)$ If $f \in \text{Fun}(S,A) \cap \text{Fun}(S,B)$

Then $\left\{ \begin{array}{l} f(a) = 0 \quad \forall a \in A \\ f(b) = 0 \quad \forall b \in B \end{array} \right.$ so $f(m) = 0 \quad \forall m \in A \cup B$

Thus $f \in \text{Fun}(S, A \cup B)$

$(\supseteq)$ If $f \in \text{Fun}(S, A \cup B)$ then $f(m) = 0 \quad \forall m \in A \cup B$
If $a \in A$ then $a \in A \cup B$ so $f(a) = 0$ so $f \in \text{Fun}(S,A)$
etc.

Ex 6. Logarithmic vector Space

$V = \{\text{Positive real numbers}\}$

Vector addition

$\underline{v} \oplus \underline{w} = v.w$

$\underline{2} \oplus \underline{3} = 2.3 = 6$

vector multi

$k.v = v^k$

$2.3 = 3^2 = 9$

$(\log 2 + \log 3 = \log 2.3 = \log 6)$ $(2.\log 3 = \log 3^2 = \log 9)$

Proof for axioms.

Chapter 4 - Subspaces

Basic Properties of Vector Subspaces

Idea Often vector spaces sit inside bigger vector spaces

→ whenever this happens the smaller V.S is like a line, or plane in the bigger one.

Def. A non-empty subset $\mathcal{U}$ of a V.S V is a vector subspace if

① closed under "+" : if $\underline{v}, \underline{w} \in \mathcal{U}$ then $\underline{v} + \underline{w} \in \mathcal{U}$

② closed under "." : If $\begin{cases} \underline{v} \in \mathcal{U} \\ k \in \mathbb{R} \end{cases}$ then $k \cdot \underline{v} \in \mathcal{U}$

Proposition. A vector subspace is a vector space contained in another vector space with the same "+" and "."

Proof: If $\mathcal{U}$ is a vector subspace then the axioms for V imply vector space axioms for $\mathcal{U}$

NOTE $\underline{0}_v \in \mathcal{U}$ because $0 \cdot \underline{v} = \underline{0}_v \in \mathcal{U}$

$\underline{0}_u$ ($\underline{0}$ vector of $\mathcal{U}$) is $\underline{0}_v$ ($\underline{0}$ vector of V)

Simply for negatives

$-v \in \mathcal{U}$ because $(-1) \cdot v = -\underline{v} \in \mathcal{U}$

Also vector spaces contained in bigger ones are clearly closed under "+" and "."

Ex. $\{(x, y, z)$ with $x + 2y - z = 0\}$ is a subspace of $\mathbb{R}^3$

Proo: If (x_1, y_1, z_1) and (x_2, y_2, z_2) satisfy $x + 2y - z = 0$ then:

$$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$$

$\Rightarrow (x_1 + x_2) + 2(y_1 + y_2) - (z_1 + z_2) = (x_1 + 2y_1 - z_1) + (x_2 + 2y_2 - z_2) = 0$ $\Rightarrow$ Closed under +

for scaling is the same.

Ex. In $\mathbb{R}^k = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} \right.$ if we fix numbers $a_1, \dots, a_k$

then the subset $\left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} \text{ with } a_1x_1 + \dots + a_kx_k = 0 \right\}$ is a Vector Space.

Def. The (linear) <u>Span</u> of a set Vectors

$$\text{Span}\{\underline{v}_1, \dots, \underline{v}_n\} = \{a_1\underline{v}_1 + \dots + a_n\underline{v}_n \text{ for any real numbers } (a_1, \dots, a_n)\}$$

Idea: Span is a "line" or "Plane" is a Vector Space.

Def. A linear Combination of vectors $\underline{v}_1, \dots, \underline{v}_n$ is something of the form:

$$a_1\underline{v}_1 + a_2\underline{v}_2 + \dots + a_n\underline{v}_n$$

Idea: A linear Combination of vectors is something on the "plane" that they determine.

Prop: Let V be a Vector space and $E \subseteq V$.

E is a subspace of V if and only if $E = \text{Span}\{E\}$
("All subspaces are "lines" or "planes"")

Proof:

($\Rightarrow$) Suppose E is a subspace if $\underline{e}_1, \dots, \underline{e}_n \in E$ then:
$a_1\underline{e}_1 + \dots + a_n\underline{e}_n \in E$ (because E is closed under Scaling and addition)
$\Rightarrow \text{Span}(E) \subseteq E$ but since $E \subseteq \text{Span}(E)$

($\Leftarrow$) Suppose $E = \text{Span}\{E\}$. If $\underline{e}_1, \underline{e}_2 \in E$ then $\underline{e}_1 + \underline{e}_2$ is a linear Combination so:

$\underline{e}_1 + \underline{e}_2 \in \text{Span}\{E\} = E$ so closed under addition

Same is the scaling. ▨

Ex. In $\mathbb{R}^3$ look at $\text{span}\{(1,0,-1),(1,1,0)\}$

$$= \{\text{Plane } x-y+z=0\}$$

$$= \text{span}\{(2,1,-1),(0,1,1)\}$$

$$= \text{span}\{(2,1,-1),(0,1,1),(0,0,0)\}$$

or $(2,2,0)$

Review

Idea: A subspace is a vector space inside a big one.

Ex. $P_k(\mathbb{R}) \subset P(\mathbb{R}) \subset Fun(\mathbb{R})$

It is easier to show something is a subspace than to show it is a vector space from scratch.

→ A subset is a subspace if

- non-empty
- closed under $+$ (if $\underline{v},\underline{w} \in S$ then $\underline{v}+\underline{w} \in S$)
- closed under $\cdot$ (if $\underline{v} \in S$ then $k.\underline{v} \in S$)

→ All supspaces are spans of some set of vectors

$$S = \text{span}\{\underline{v}_1, \dots, \underline{v}_n\} = \{a_1\underline{v}_1 + \dots + a_n\underline{v}_n\}$$

* All spans are subspaces.

Note. Since subspaces are vector spaces, every subspace contains $\underline{0}$.

→ if $\underline{0} \notin S$ then S is not a subspace.

Two basic subspace operations:

Intersection

Prop: If S, T are subspace, so is $S \cap T$

Proof: . ($S \cap T \neq 0$) if S, T subspaces, $\underline{0} \in S$ and T

So $\underline{0} \in S \cap T$

$\Rightarrow S \cap T \neq 0$

. ($S \cap T$ closed under $+$) If $\underline{v}, \underline{w} \in S \cap T$ $\begin{cases} \underline{v}, \underline{w} \in S \Rightarrow \underline{v}+\underline{w} \in S \\ \underline{v}, \underline{w} \in T \Rightarrow \underline{v}+\underline{w} \in T \end{cases}$

$\Rightarrow \underline{v}+\underline{w} \in S \cap T$

. ($S \cap T$ closed under scaling) Same ▨

Ex. Line on $\mathbb{R}^3$, $F(t) = t\langle 1,2,3 \rangle$

$x = t$

$y = 2t$

$z = 3t$

↳ symmetric eqn.

$\Rightarrow x = \frac{1}{2} y = \frac{1}{3} z$

$\begin{cases} x = \frac{1}{2} y \\ x = \frac{1}{3} z \end{cases}$

$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ where } x = \frac{1}{2} y \right\} \cap \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ where } x = \frac{1}{3} y \right\}$

plane $\cap$ plane

Sum

Def. If S, T are subspaces then

$$S+T = \{ \underline{s}+\underline{t} \text{ where } \underline{s} \in S, \ \underline{t} \in T \}$$

Prop. If S, T subspaces, so is $S+T$

Proof. . $\underline{0} \in S, T \subseteq \underline{0}+\underline{0} = \underline{0} \in S+T$

$\Rightarrow S+T \neq 0$

. If $\underline{v}, \underline{w} \in S+T$ then $\underline{v} = \underline{s}_1 + \underline{t}_1$, $\underline{w} = \underline{s}_2 + \underline{t}_2$

So $\underline{v}+\underline{w} = (\underline{s}_1+\underline{t}_1) + (\underline{s}_2+\underline{t}_2) = (\underline{s}_1+\underline{s}_2) + (\underline{t}_1+\underline{t}_2)$

$\Rightarrow \underline{v}+\underline{w} \in S+T$.

. Same for Scaling ▨

Properties:

Prop: $S+T = \text{Span}\{S \cup T\}$ (exercise 1b)

Prop: $S+T$ is the smallest subspace containing S and T (exercise 1a)

Ex from 120: A plane is determined by summation of two lines.

line + line = plane they determine.

Three Examples of subspaces (of $\mathbb{R}^n$, $P_k(\mathbb{R})$, $\text{Fcn}(S)$)

Ex. $\mathcal{U} \subset \mathbb{R}^n$ where $\mathcal{U} = \left\{ \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \text{ with } a_1 = a_2 = \cdots = a_n \right\}$

- $\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}\right] \in \mathcal{U}$ because $0 = 0 = \cdots = 0$,
  So $\mathcal{U} \neq 0$
- $\begin{bmatrix} a \\ \vdots \\ a \end{bmatrix} + \begin{bmatrix} b \\ \vdots \\ b \end{bmatrix} = \begin{bmatrix} a+b \\ \vdots \\ a+b \end{bmatrix} \in \mathcal{U}$ because $a+b = \cdots = a+b$
- $k \cdot \begin{bmatrix} a \\ \vdots \\ a \end{bmatrix} = \begin{bmatrix} ka \\ \vdots \\ ka \end{bmatrix} \in \mathcal{U}$ because $ka = \cdots = ka$

Note: $\mathcal{U} = \text{Span}\left\{ \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \right\}$

$$\mathcal{U} = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \text{ with } a_1 = a_2 \right\} \cap \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \text{ with } a_2 = a_3 \right\} \cap \cdots \cap \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \text{ with } a_{n-1} = a_n \right\}$$

Ex. $Z \subset P_k(\mathbb{R})$ where $Z = \{a_0 + a_1 x + \cdots + a_k x^k \text{ with } a_0 = 0\}$

- Zero polynomial $\underline{0} = 0 + 0x + \cdots + 0x^k \in Z$ because constant terms are zero.
  Two way of think of this:
  1) First coefficient is zero $\Longleftrightarrow$ $P = x \cdot q$
  2) evaluating polynomial at 0 gives 0 $\Longleftrightarrow$ $p(0) = 0$

- $P_1, P_2 \in Z$ then constant coeff. $= 0$ so constant coeff. of $P_1 + P_2 = 0 + 0$ thus $P_1 + P_2 \in Z$
- Scaling is the same. $\boxed{/\!/\!/}$

Ex. $N \subset Fun(S)$ by $N = \left\{ f : S \to \mathbb{R} \text{ with } f(a) + f(b) = 0 \right\}$
$(a, b \in S)$

. The Zero function $\underline{0}(s) = 0$ has $\underline{0}(a) + \underline{0}(b) = 0 + 0$

So $\underline{0} \in N$

$\Rightarrow N \neq 0$

. If $f, g \in N$ then $f(a) + f(b) = 0$

$g(a) + g(b) = 0$

So $(f+g)(a) + (f+g)(b)$

$= (f(a) + g(a)) + (f(b) + g(b))$

$= (f(a) + f(b)) + (g(a) + g(b))$

$= 0 + 0$

$= 0.$

$\Rightarrow f+g \in N$

. Closed under Scaling is the same.

Prop. Let V be a vector space. Suppose $\mathcal{U}$ and W are subspaces of V the $\mathcal{U} \cap W$ is also a subspace of V.

Remark: $\mathcal{U} \cup W$ is in general <u>not</u> a subspace.

Proof: ① Since $\mathcal{U}, W$ are subspaces, $\vec{0} \in \mathcal{U}$ and $\vec{0} \in W$

$\Rightarrow \vec{0} \in \mathcal{U}, W \Rightarrow \mathcal{U} \cap W \neq \emptyset$

Recall: Say V is a vector space and $W \subset V$

Thm $\underline{W}$ is a subspace of V if and only if:

① $W \neq 0$

② If $v, w \in W \Rightarrow v + w \in W$

③ If $c \in \mathbb{R}, v \in W \Rightarrow c \cdot v \in W$

② Suppose $v, w \in \mathcal{U} \cap W$. Is $v + w \in \mathcal{U} \cap W$?

$v, w \in \mathcal{U} \cap W \Rightarrow v, w \in V$ and $v, w \in W$

$\Rightarrow v + w \in \mathcal{U} \cap W$

$\left\{ \begin{array}{l} \mathcal{U} \text{ is subspace} \\ W \sim \quad \sim \end{array} \right.$

③ Suppose $c \in \mathbb{R}, v \in \mathcal{U} \cap W$. Is $cv \in \mathcal{U} \cap W$?

$v \in \mathcal{U} \cap W \Rightarrow v \in \mathcal{U}$ and $v \in W$ $\underline{\Rightarrow cv \in \mathcal{U} \text{ \& } cv \in W}$

$\Rightarrow cv \in \mathcal{U} \cap W$ ✓

$\Rightarrow$ 3 conditions are o.k.

$\Rightarrow \mathcal{U} \cap W$ is a subspace.

Def. Say V is a vector space and $v_1, v_2, \dots, v_n \in V$. Say $c_1, c_2, \dots, c_n \in \mathbb{R}$. A vector of the form

$c_1v_1 + c_2v_2 + \dots + c_nv_n$ is called a linear combination of $v_1, \dots, v_n$.

Ex. Let $V = \mathbb{R}^3$

$v_1 = (1,1,0)$

$v_2 = (0,-5,0)$

Is $u = (0,0,2)$ a linear combination of v_1, v_2?

Is $w = (1,-1,0)$ a linear combination of v_1, v_2?

① $u = c_1v_1 + c_2v_2$

$$\left.\begin{array}{l}(0,0,2) = c_1(1,1,0) + c_2(0,-5,0) \\ (0,0,2) = (c_1, c_1 - 5c_2, 0)\end{array}\right\} \Rightarrow \equiv \begin{array}{l} 0 = c_1 \\ 0 = c_1 - 5c_2 \\ \times\ 2 = 0 \end{array}$$

② $w = c_1v_1 + c_2v_2$

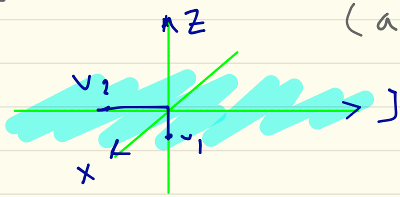
$$\left.\begin{array}{l}(1,-1,0) = c_1(1,1,0) + c_2(0,-5,0) \\ (1,-1,0) = (c_1, c_1 - 5c_2, 0)\end{array}\right\} \equiv \begin{array}{l} 1 = c_1 \\ -1 = c_1 - 5c_2 \\ 0 = 0 \end{array} \Rightarrow \begin{cases} c_1 = 1 \\ c_2 = 2/5 \end{cases} \checkmark$$

Describe all linear combinations of v_1, v_2:

$(a,b,c) = c_1(1,1,0) + c_2(0,-5,0)$

$$\left.\begin{array}{l} a = c_1 \\ b = c_1 - 5c_2 \\ c = 0 \end{array}\right\} \begin{array}{l} \boxed{c_1 = a} \\ \boxed{c_2 = \frac{a-b}{5}} \\ \boxed{c = 0} \end{array}$$

$\Rightarrow$ all linear combinations: vectors of the form $(a,b,0)$



Def. Say V is a vector space & $v_1, v_2, \dots, v_n \in V$

The <u>Span</u> of $v_1, \dots, v_n$ is the set of all linear combinations of $v_1, \dots, v_n$ $\quad \text{Span}(v_1, \dots, v_n)$

(In the example before

$$\text{Span}(v_1, v_2) = \{ (a, b, 0) \mid a, b \in \mathbb{R} \}$$

Thm: Say V is a vector space & $v_1, v_2, \dots, v_n \in V$ then, $\text{Span}(v_1, \dots, v_n) = W$ is a subspace of V.

Proof: (1) $W \neq 0$ take $c_1 = c_2 = \dots = c_n = 0$

$$0v_1 + 0v_2 + \dots + 0v_n = \vec{0} \in W \Rightarrow W \neq \emptyset$$

(2) Say $u, w \in W$ Is $u + v \in W$?

$u = c_1 v_1 + \dots + c_n v_n$ for some $c_1, \dots, c_n$

$v = d_1 v_1 + \dots + d_n v_n$

$u + v = (c_1 + d_1) v_1 + \dots + (c_n + d_n) v_n$ is a Linear Comb.

$\Rightarrow u + v \in W$

(3) Say $v \in W$, $c \in \mathbb{R}$, Is $c \cdot v \in W$?

$v = c_1 v_1 + \dots + c_n v_n$

$c \cdot v = (cc_1) v_1 + (cc_2) v_2 + \dots + (cc_n) v_n$

$cv \in W$

Bonus: Ch 4. Ex 19

Prove: $\text{Span}(E) = \bigcap(\text{subspaces containing } E)$

(→ This means Span(E) is the smallest subspace containing E)

Proof:

($\supseteq$) Note that Span(E) is a subspace containing E

$$\text{Span}(E) \supseteq \cdots \cap \text{Span}(E) \cap \cdots = \bigcap(\text{all subspaces containing } E)$$

($\subseteq$) If $\underline{v} \in \text{Span}(E)$ then $\underline{v} = a_1\underline{e_1} + \cdots + a_n\underline{e_n}$ where $\underline{e_i} \in E$

If V is a subspace containing E then $\underline{e_i} \in V$

V is closed under scaling so $a_i\underline{e_i} \in V$

V is closed under addition so $a_1\underline{e_1} + \cdots + a_n\underline{e_n} \in V$

Thus $\underline{v} \in V$ for every subspace V with $E \subseteq V$

Therefore $\underline{v} \in \bigcap(\text{all subspace containing } E)$

Bonus. Ch 4 Ex 29

Proof: If $\underline{u} + \underline{v} + \underline{w} = 0$ then $\text{Span}(\underline{u}, \underline{v}) = \text{Span}(\underline{v}, \underline{w})$

Proof:

($\subseteq$) ($\underline{u} = -\underline{v} - \underline{w}$) Let $\underline{x} \in \text{Span}(\underline{u}, \underline{v})$

we want to show $\underline{x} \in \text{Span}(\underline{v}, \underline{w})$

Then $\underline{x} = a \cdot \underline{u} + b \cdot \underline{v}$

$= a(-\underline{v} - \underline{w}) + b \cdot \underline{v}$

$= (b-a) \cdot \underline{v} + (-a) \cdot \underline{w}$

So $\underline{x} \in \text{Span}(\underline{v}, \underline{w})$

1, 4, 7, 11, 13, 16, 17, 19, 24, 26, 29, 34

Chapter 5: Linear Independence and Independence

Idea: In bonus Problem

$\underline{u} + \underline{v} + \underline{w} = 0$

$\text{Span}(\underline{u}, \underline{v}, \underline{w}) = \text{Span}(\underline{u}, \underline{v}) = \text{Span}(\underline{v}, \underline{w})$

Basic Def:

A set E is linearly dependent

If there are $\underline{e}_1, \dots, \underline{e}_n \in E$ with $a_1\underline{e}_1 + \dots + a_n\underline{e}_n = 0$ where some $a_i \neq 0$

The formula $a_1\underline{e}_1 + \dots + a_n\underline{e}_n = 0$ is a "linear (dependence) relation"

Ex. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is linearly dependence (set)

because $1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \underline{0}$

Ex. $\{ P(x) \text{ where } \deg(p) \leq 1 \}$ is dependent set

because

$1 \cdot 1 + 1 \cdot \underline{x} + (-1)(1+x) = \underline{0}$

Def:

A set E is linearly independent

if it is not linearly dependent

Ex. $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is an independent Set.

Proof: Suppose $a\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$

(want to show $a, b, c = 0$ is only solution)

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ a \\ a \end{bmatrix} + \begin{bmatrix} 0 \\ b \\ b \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix} = \begin{bmatrix} a \\ a+b \\ a+b+c \end{bmatrix}$$

So $a = 0$

$a+b = 0 \Rightarrow b = 0$

$a+b+c = 0 \Rightarrow c = 0$

Ex. $\{1+x, 1-x\} \subset P(\mathbb{R})$ is lin. Ind.

Proof: Suppose $a(1+x)+b(1-x)=0$

$$(a+b)+(a-b)x = 0$$

So $a+b=0$

$\oplus$ $a-b=0$

$$2a = 0 \longrightarrow a=0$$

$$a+b=0 \Longrightarrow b=0$$

Ex. Define $\chi_t(s) = \begin{cases} 1 & \text{if } s=t \\ 0 & \text{if } s\neq t \end{cases}$ characteristic function

$\{\chi_t\} \subset Fun(S)$ is Ind.

Proof: Suppose $a_1\chi_{t_1} + \cdots + a_n\chi_{t_n} = 0$

then $(a_1\chi_{t_1} + \cdots + a_n\chi_{t_n})(t_1) = 0$

$a_1\chi_{t_1} + \cdots + a_n\chi_{t_n}(t_n) =$

$a_1 \cdot 1 + a_2 \cdot 0 + \cdots + a_n \cdot 0 =$

$a_1 =$

Similarly all other $a_i = 0$ $\blacksquare$

Bonus: Ch 5 Ex 1

Dep. / Indep.

$\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right\} \rightarrow 1\begin{bmatrix}1\\1\\1\end{bmatrix} + (-1)\begin{bmatrix}0\\1\\0\end{bmatrix} + (-1)\begin{bmatrix}1\\0\\1\end{bmatrix} = \underline{0}$ Dep

$\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}\right\} \rightarrow$ Ind $\begin{cases} a=0 \\ b=0 \\ c=0 \end{cases}$

$\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\} \rightarrow$ $\begin{matrix} a+c=0 \\ b+c=0 \\ c=0 \end{matrix}$ ind

$\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}\right\} \rightarrow a=0, b=0,$ ind.

Ex 2 (more likely as exam Prob)

$\{1, x, x^2\}$ — ind. $a \cdot 1 + b \cdot x + c \cdot x^2 = 0$
$a, b, c = 0$

$\{1+x, 1-x, x^2, 1\}$ — dep. $(1 \cdot (1+x) + 1(1-x) + 0(x^2) + (-2)(1) = 0)$

$\{x^2-1, x+1, x^2-x, x^2+x\}$ — dep

$\{x-x^2, x^2-x\}$ — dep

$\{1, 1-x, 1-x^2\}$ — ind.

Properties of Dep. & Idep. Sets

Goal: Take dep. Set and an Indep. Set out of it.

Prop: If $\underline{0} \in E$ then E is a dep set.

Proof:

$1 \cdot 0 = \underline{0}$ (one $\underline{0}$ of everything else) is a dep. relation.

Corollary: Subspaces are dep. Set

recall: $S = \text{Span}(S)$ if S is a Subspace

→ Cor. means $S = \text{Span}\{\text{sth smaller than } S\}$

what to write: $S = \text{Span}\{\text{Very few vectors}\}$

Notation:

A vector $\underline{w}$ is *linearly dependent* on a set E if $\underline{w} \in \text{Span}(E)$

(i.e $\underline{w} = a_1 \underline{e_1} + \cdots + a_n \underline{e_n}$ where $\underline{e_i} \in E_i$)

Prop:

E is a dep. set if and only if there is a $\underline{v} \in E$ with $\underline{v}$ dep. on $E_{\underline{v}}$

Proof:

E is dep means there is $\underline{v}_1, \dots, \underline{v}_n \in E$ with $a_1\underline{v}_1 + \dots + a_n v_n = 0$
$a_1 \neq 0$

$$\underline{v}_1 = -\frac{a_2}{a_1}\underline{v}_2 - \frac{a_3}{a_1}\underline{v}_3 - \dots - \frac{a_n}{a_1}\underline{v}_n \quad (\text{Solve for } \underline{v}_1)$$

So $\underline{v}_1$ is dep. on $E_{\underline{v}_1}$

($\Leftarrow$) If $\underline{v}$ is dep. on $E \setminus \underline{v}$ then $\underline{v} = a_1\underline{e}_1 + \dots + a_n\underline{e}_n$

So $0 = a_1\underline{e}_1 + \dots + a_n\underline{e}_n + (-1)\underline{v}_1$

Important Idea:

Use dependence relation to solve for one vector in terms of others.

Thm:

If E is a finite dep set $(E \neq \{\underline{0}\})$ then there is a subset $F \subset E$ with

① F is indep.

② $\mathrm{Span}(F) = \mathrm{Span}(E)$

Remark:

If E is indep. then removing any element makes span smaller.
(i.e $\mathrm{Span}(F) \neq \mathrm{Span}(E)$ if $F \subsetneq E$)

Proof: If E is indep. set, then there is $\underline{v} \in E$ with $\underline{v}$ dep on $E \setminus \underline{v}$ Thus

$$\underline{v} = a_1\underline{e}_1 + \dots + a_n\underline{e}_n \quad \text{where } \underline{e}_i \in E \setminus \underline{v}$$

Claim:

($\supseteq$) $E \setminus \underline{v} \subset E$ So $\mathrm{Span}(E \setminus \underline{v}) \subset \mathrm{Span}(E)$

$(\subseteq)$ If $\underline{w} \in \mathrm{Span}(E)$

$$\underline{w} = b_1 \underline{e_1} + \cdots + b_n \underline{e_n} + K \cdot \underline{v}$$
$$= (a_1 + b_1)\underline{e_1} + \cdots + (a_n + b_n)\underline{e_n}$$

So $\underline{w} \in \mathrm{Span}(E \setminus \underline{v})$

Use this method to shrink E without changing the span.
Since E is finite eventually we must have an indep. set
(Note : $F = \{v\}$ is indep. if $\underline{v} \neq \underline{0}$)

Note: Proof of thm. gave a method for finding indep. subsets.

Ex (exercise 6)
Find an indep. subset of
$\{x^3, x^3 - x^2, x^3 + x^2, x^2 - 1\}$ with the same span.

Dep. relation $x^3 = \frac{1}{2}(x^3 - x^2) + \frac{1}{2}(x^3 + x^2)$ $\longrightarrow$ throw away x^3 $\{x^3 - x^2, x^3 + x^2, x^3 - 1\}$ indep

Ex. $\left\{ \cancel{\begin{bmatrix}1\\0\\0\end{bmatrix}}, \cancel{\begin{bmatrix}0\\1\\0\end{bmatrix}}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix} \right\}$

$\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}0\\1\\0\end{bmatrix} - \begin{bmatrix}0\\0\\1\end{bmatrix} \Rightarrow \begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}1\\0\\1\end{bmatrix}$ indep

Ex: If S is a finite set $\{\chi_t\}_{t \in S}$ is indep. set
and $\mathrm{Span}(\{\chi_t\}) = \mathrm{Fun}(S)$

Proof: If $f \in \mathrm{Fun}(S)$ then $f(s) = f(t_1) \cdot \chi_{t_1}(s) + f(t_2)\chi_{t_2}(s) + \cdots + f(t_n)\chi_{t_n}(s)$
$(S = \{t_1, t_2, \ldots, t_n\}$
$= (f(t_1) \cdot \chi_{t_1} + \cdots + f(t_n)\chi_{t_n})(s)$

Important Exercises: 9, 10, 15, 17, 18, 21, 26

Important Exercises from chapter 5:

(15) Prove: E linearly independent if and only if

$$\mathrm{Span}(F) \neq \mathrm{Span}(E) \text{ for all } F \subsetneq E$$

(17) Prove: If E, F independent then

I) $E \cap F$ independent

II) $E \cup F$ independent if and only if

(18) & (21) $\mathrm{Span}(E) \cap \mathrm{Span}(F) = \{\underline{0}\}$

(9) & (10) If $E \subset F$, F indep then E indep

If $E \subset F$, E dep. Then F dep.

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chapter 3

(12) (check Ax 1,3,4,8)

chapter 4

(6)

Chapter 6.

Finite-Dimensional Vector Spaces and Bases.

Recall: $\text{Span}\{\underline{e_1}, \underline{e_2}, \ldots, \underline{e_n}\} = \{a_1\underline{e_1}, a_2\underline{e_2}, \ldots, a_n\underline{e_n}\}$

to show $\text{Span}\{\underline{e_1}, \underline{e_2}, \ldots, \underline{e_n}\} = V$

Every vector $\underline{v} \in V$ can be written as.

$$\underline{v} = a_1\underline{e_1} + a_2\underline{e_2} + \cdots + a_n e_n \quad \left(\begin{matrix}\text{solve for} \\ a_1, a_2, \cdots\end{matrix}\right)$$

- $\{\underline{e_1}, \ldots, \underline{e_n}\}$ is independent if

$$0 = a_1\underline{e_1} + a_2\underline{e_2} + \cdots + a_n e_n$$

has only the solution $\begin{cases} a_1 = 0 \\ a_2 = 0 \\ \vdots \\ a_n = 0 \end{cases}$

NOTE: If $\{\underline{e_1}, \ldots, \underline{e_n}\}$ spans V and is independent then

$$\underline{v} = a_1\underline{e_1} + \cdots + a_n\underline{e_n}$$

has exactly one solution for all $\underline{v} \in V$

($(a_1, a_2, \ldots, a_n)$ is "coordinates of $\underline{v}$" with respect to $\{\underline{e_1}, \ldots e_n\}$)

Finite Dim. Vector Spaces

Def.

A vector space V is finite dimensional

if there is a finite set E with $\text{span}(E) = V$.

Basic Examples: $\mathbb{R}^k$, $P_k(\mathbb{R})$, $Fun(S)$ if S is finite.

Ex 1. $\mathbb{R}^k$ is finite dimensional
("standard basis")

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\}$$

Ex. in $\mathbb{R}^3$:

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

B spans $\mathbb{R}^k$ because

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \dots + x_k \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Ex 2. $P_k(\mathbb{R})$ is finite dimensional
("standard basis")

$$B = \{ \underline{1}, \underline{x}, \underline{x^2}, \dots, \underline{x^k} \}$$

- $\mathrm{Span}(B) = P_k(\mathbb{R})$ because

$$a_0 + a_1 x + a_2 x^2 + \dots + a_k x^k = a_0 \cdot \underline{1} + a_1 \cdot \underline{x} + a_2 \cdot \underline{x^2} + \dots + a_k \cdot \underline{x^k}$$

Side Note

often we identify

$P_k(\mathbb{R}) \subseteq \mathbb{R}^{k+1}$

$$a_0 + a_1 x + \dots + a_k x^k \longleftrightarrow \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_k \end{bmatrix}$$

Ex3. Fun(S), where S is finite, is finite dimensional ("standard basis")

$$B = \{\chi_s\}_{s \in S}$$

characteristic function

Def. B is a basis for the vector space V if

① $\text{Span}(B) = V$ ← B is big

② B is linear indipendent ← B is not too big

Thm. If V is finite dimensional, then V has a basis.

Proof.

If V is f.d then there is a finite set E with $\text{span}(E) = V$ (by def).

from thm last time E has a subset F with F linearly ind. and $\text{Span}(F) = \text{Span}(E) = V$

Let $B = F$ ▨

In Previous examples, our spanning sets were basis.

Ex1 (revisited).

. B is linearly ind: if $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + a_k \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix}$ So $\begin{cases} a_1 = 0 \\ a_2 = 0 \\ \vdots \\ a_k = 0 \end{cases}$

Ex. $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right\}$ is a basis for $\mathbb{R}^2$:

- Span $= \mathbb{R}^2$:

scratch work

$$\begin{bmatrix}x\\y\end{bmatrix} = a\begin{bmatrix}1\\1\end{bmatrix} + b\begin{bmatrix}1\\0\end{bmatrix}$$ Solve for a & b

$$\Rightarrow \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}a+b\\a\end{bmatrix} \rightsquigarrow \begin{cases} a = y \\ b = x-y \end{cases}$$

$$\Rightarrow \begin{bmatrix}x\\y\end{bmatrix} = y\begin{bmatrix}1\\1\end{bmatrix} + (x-y)\begin{bmatrix}1\\0\end{bmatrix}$$

- Linearly independent.

$$\text{If } \begin{bmatrix}0\\0\end{bmatrix} = a\begin{bmatrix}1\\1\end{bmatrix} + b\begin{bmatrix}1\\0\end{bmatrix}$$

$$= \begin{bmatrix}a+b\\a\end{bmatrix} \rightsquigarrow \begin{cases} 0 = a \\ 0 = b \end{cases} \quad \blacksquare$$

Ex. $\left\{\underline{1}, \underline{x-1}, \underline{(x-2)(x-1)}\right\}$ is a basis for $P_2(\mathbb{R})$

- Span $= P_2(\mathbb{R})$:

solve for a_1, a_2, a_3

Scratch work: $b_0 + b_1 x + b_2 x^2 = a_1 \cdot 1 + a_2 (x-1) + a_3 \cdot \underline{(x-2)}\,\underline{(x-1)}$

$$\Rightarrow b_0 + b_1 x + b_2 x^2 = a_1 + a_2 x - a_2 + a_3 x^2 - 3a_3 x + 2a_3$$

$$= (a_1 - a_2 + 2a_3) + (a_2 - 3a_3)x + a_3 x^2$$

$$\Rightarrow \begin{cases} b_0 = a_1 - a_2 + 2a_3 \\ b_1 = a_2 - 3a_3 \\ b_2 = a_3 \end{cases} \longrightarrow \begin{cases} a_3 = b_2 \\ a_2 = b_1 + 3b_2 \\ a_1 = b_0 + b_1 + b_2 \end{cases}$$

$$\Rightarrow b_0 + b_1 x + b_2 x^2 = (b_0 + b_1 + b_2) \cdot \underline{1} + (b_1 + 3b_2) \cdot \underline{(x-1)} + b_2 \underline{(x-2)(x-1)}$$

- Indep. (same as before)

$$0 = \ldots \Rightarrow \begin{cases} a_3 = b_2 = 0 \\ a_2 = b_1 + 3b_2 = 0 \\ a_1 = b_0 + b_1 + b_2 = 0 \end{cases} \Rightarrow \begin{cases} b_0 = 0 \\ b_1 = 0 \\ b_2 = 0 \end{cases} \quad \blacksquare \checkmark$$

Properties of Bases (Dimensions)

Prop: Let $E = \{\underline{e}_1, \dots, \underline{e}_k\}$ Then E is linearly dependent if and only if

Some $\underline{e}_m$ has $\underline{e}_m \in \mathrm{Span}\{\underline{e}_1, \dots, \underline{e}_{m-1}\}$

(i.e Some vectors is dependent on previous ones)

Proof:

($\Leftarrow$) If $\underline{e}_n \in \mathrm{Span}\{\underline{e}_1, \dots, \underline{e}_{m-1}\}$ then

$$\mathrm{Span}\{\underline{e}_1, \dots, \underline{e}_{m-1}\} \subset \mathrm{Span}\{\underline{e}_1, \dots, \underline{e}_{m-1}, \underline{e}_{m+1}, \dots, \underline{e}_k\}$$

So E is linearly dependent.

($\Rightarrow$) If E is linearly dependent then

$$a_1\underline{e}_1 + \dots + a_k\underline{e}_h = \underline{0} \quad \text{for some } a_i.$$

Let $m =$ largest, so that $a_i \neq 0$, solve for $\underline{e}_m$:

$$\underline{e}_m = -\frac{a_1}{a_m}\underline{e}_1 - \dots - \frac{a_{m-1}}{a_m}\underline{e}_{m-1} + \underbrace{0 \dots}_{\text{all higher } a_i = 0}$$

So $\underline{e}_m \in \mathrm{Span}\{\underline{e}_1, \dots, \underline{e}_{m-1}\}$ ▨

Prop: If $E \subset \mathrm{Span}\{\underline{v}_1, \dots, \underline{v}_n\}$ is linearly indep. Then

$$\#\,\text{elements}(E) \leq n$$

Proof: (To deal with infinite E: let $F \subset E$ be finite.)

we will work with F.

$F = \{\underline{e}_1, \dots, \underline{e}_k\}$ (Goal: show $k \leq n$.

Plan: Add $\underline{e}_i$ to $\{\underline{v}_1, \dots, \underline{v}_n\}$ one at a time

show that each time we add a $\underline{e}_i$ we can

remove a $\underline{v}_i$ keeping same span.)

$\{\underline{e}_1, \underline{v}_1, \dots, \underline{v}_n\}$ is linearly dep. because

$\underline{e}_i \in \mathrm{Span}\{\underline{v}_1, \dots, \underline{v}_n\}$

So some vector is dependent on previous ones.

NOTE: Not $\underline{e}_1$ because $\underline{e}_1 \neq 0$

without loss of generality dependent vector is $\underline{v}_n$.

NOTE : $\text{Span}\{\underline{e}_1, \underline{v}_1, \dots, \underline{v}_{n-1}\} = \text{Span}\{\underline{e}_1, \underline{v}_1, \dots, \underline{v}_n\} = \text{Span}\{\underline{v}_1, \dots, \underline{v}_n\}$

$\Rightarrow \{\underline{e}_2, \underline{e}_1, \underline{v}_1, \dots, \underline{v}_{n-1}\}$ is linearly dep. because

$\underline{e}_2 \in \text{Span}\{\underline{v}_1, \dots, \underline{v}_n\} = \text{Span}\{\underline{e}_1, \underline{v}_1, \dots, \underline{v}_{n-1}\}$

So some vector is dependent on previous ones.

Ch5 Ex 10:

If F is indep. $E \subset F$

Then E is indep.

NOTE: not $\underline{e}_2$ because $\underline{e}_2 \neq 0$

Not $\underline{e}_1$ because $\{\underline{e}_1, \underline{e}_2\}$ linearly indep.

If $k > n$ then repeating n times gives

$\text{Span}\{\underline{e}_k, \underline{e}_{k-1}, \dots, \underline{e}_1\}$ (add $n e_i$ remove all $\underline{v}_i$)

$\text{Span}\{\underline{v}_1, \dots, \underline{v}_n\}$

But $\underline{e}_k \in \text{Span}\{\underline{v}_1, \dots, \underline{v}_n\} = \text{Span}\{\underline{e}_n, \dots, \underline{e}_1\}$

This would mean that F was not independent #

Thus $k \leq n$ ∎

Thm: Every finite dimensional vector space has a basis (existence of bases)

Furthermore all basis of a vector space have the same number of elements. (Uniqueness of basis)

(The "Dimension" of V)

Proof: Existence was last time

For uniqueness: If $E = \{\underline{e}_1, \dots, \underline{e}_n\}$ are both bases of V.

Rank Thm

Recall !!!

$E \subset \text{span}(F) = V$ so $n \leq m$

$F \subset \text{Span}(E) = V$ so $m \leq n$ ∎

Ex. $P(\mathbb{R})$ is not finite dimensional.

Proof: $\{1, x, x^2, \dots\} \subset P(\mathbb{R})$ is an infinite indep. set

If $P(\mathbb{R})$ was finite dim. then

There would be basis $B = \{\underline{b}_1, \dots, \underline{b}_n\}$

Any indep. set

$E \subset P(\mathbb{R}) = \text{Span}\{b_1, \dots, b_n\}$ would have $\leq n$ elements.

Thm. Finite dimensional $\iff$ linear Indep. Sets all finite.

Infinite dimensional $\iff$ there is an infinite linear indep. Set.

Proof: Previous

($\Leftarrow$) Example argument does this.

($\Rightarrow$) If V is not finite dimensional then no finite set spans it.

Pick $\underline{e}_1 \neq \underline{0}$ in V. Span$\{\underline{e}_1\} \neq V$

Pick $\underline{e}_2 \in V \setminus$ Span$\{\underline{e}_1\}$ then $\{\underline{e}_1, \underline{e}_2\}$ indep. Also Span$\{\underline{e}_1, \underline{e}_2\} \neq V$

Pick $\underline{e}_3 \in V \setminus$ Span$\{\underline{e}_1, \underline{e}_2\}$ then $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ indep. Also Span$\{\underline{e}_1, \underline{e}_2, \underline{e}_3\} \neq V$

etc.

This builds an infinite independent set ∎

Def: (Let V be finite dimensional)

The dimensions of V "$\dim(V)$" is the member of elements in a basis for V.

If V is infinite dimensional, Say $\dim(V) = \infty$

Ex. $\dim(P(\mathbb{R})) = \infty$

$\dim(\mathrm{Fun}(S)) = \infty$ if S is infinite

$\dim(\mathbb{R}^k) = k$ basis $= \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$

$\dim(P_k(\mathbb{R})) = k+1$ basis $= \{1, x, \ldots, x^k\}$

Outline for 6.3 "Using Basis"

Idea:

If V has basis $B = \{\underline{b}_1, \ldots, \underline{b}_n\}$ then we can write

$$\underline{v} = a_1 \underline{b}_1 + \cdots + a_n \underline{b}_n \qquad \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n$$

Unique.

"Coordinates of $\vec{v}$ relative to Basis B"

→ Replace Vector space by $\mathbb{R}^n$.

Thm. If $B = \{\underline{b}_1, \dots, \underline{b}_n\}$ is a basis then $\underline{v} = a_1\underline{b}_1 + \dots + a_n\underline{b}_n$ Uniquely for all $\underline{v} \in V$ (Uniqueness and Existence of Coordinates)

$\longrightarrow$ You can remove elements from spanning set to get a basis.

Thm. Independent sets can be extended to get a basis.
(If E has k elements, $E \subset V$ with $\dim n$ then there is set F with $n-k$ elements so that $E \cup F$ is a basis for V) Baisis Extention Thm.

$\Rightarrow$ If $U \subset V$ subspace then
$$\dim(U) \leq \dim(V)$$

Thm. If $\{e_1, \dots, e_n\} \subset V$ and $\dim(V) = n$

- If $\{\underline{e}_1, \dots, \underline{e}_n\}$ indep then it is a basis for V.
- If $\mathrm{Span}\{e_1, \dots, e_n\}$ then it is a basis for V.

Ex. $\{1, x, x^2 + 2x - 1, 2x^2 - x\}$ is linearly dependent

$\cap$

$P_2(\mathbb{R})$ with $\dim = 3$ so set of 4 vectors can not be indep.

Ex. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix} \right\}$ is linealy dep.

$\cap$

$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ with } x - y - z = 0 \right\} \leftarrow \dim = 2$

Chapter 7. Numerical (Computational Examples)

Ex: Is $\begin{bmatrix}1\\-2\\0\\3\end{bmatrix}$ a linear comb. of $\left\{\begin{bmatrix}3\\9\\-4\\-2\end{bmatrix}, \begin{bmatrix}2\\3\\0\\-1\end{bmatrix}, \begin{bmatrix}2\\-1\\2\\1\end{bmatrix}\right\}$?

Solve:

$$\begin{bmatrix}1\\-2\\0\\3\end{bmatrix} = a\begin{bmatrix}3\\9\\-4\\-2\end{bmatrix} + b\begin{bmatrix}2\\3\\0\\-1\end{bmatrix} + c\begin{bmatrix}2\\-1\\2\\1\end{bmatrix}$$

$$\begin{cases} 1 = 3a + 2b + 2c & (1) \\ -2 = 9a + 3b - c & (2) \\ 0 = -4a \quad + 2c & (3) \\ 3 = -2a - b + c & (4) \end{cases}$$

Need a solution to all 4 equations.

$(1)\quad 1 = 3a + 2b + 2c$

$(4)\quad 2(3 = -2a - b + c)$

$7 = -a \quad + 4c$

$(3)\quad -2(0 = -4a \quad + 2c)$

$7 = 7a \implies a = 1,\ c = 2,\ b = -3$

Solution to (2)

$-2 \overset{\checkmark}{=} 9(1) + 3(-3) - 2$

$\xRightarrow{\text{Yes}}$ $\begin{bmatrix}1\\-2\\0\\3\end{bmatrix} = 1\begin{bmatrix}3\\9\\-4\\-2\end{bmatrix} - 3\begin{bmatrix}2\\3\\0\\-1\end{bmatrix} + 2\begin{bmatrix}2\\-1\\2\\1\end{bmatrix}$

Ex2. Is $(1+m - 2m^2 + 4m^3)$ in Span $\{1-m,\ 1-m^2,\ 1-m^3\}$

$$(1+m-2m^2+4m^3) = a(1-m) + b(1-m^2) + c(1-m^3)$$
$$= (a+b+c) + (-a)m + (-b)m^2 + (-c)m^3$$

$1 = a+b+c$

$$\left.\begin{array}{ll} (m) & 1 = -a \\ (m^2) & -2 = -b \\ (m^3) & 4 = -c \end{array}\right\} \quad \begin{array}{l} a = -1 \\ b = 2 \\ c = -4 \end{array}$$

But $1 \neq (-1) + 2 - 4$

No

Ex. $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \text{ with } x_1 - x_2 + x_3 - x_4 = 0 \right\} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_1 - x_2 + x_3 \end{bmatrix} \right\}$

Standard Basis
$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

$T = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \text{ with } x_1 + x_2 + x_3 + x_4 = 0 \right\}$

Find a basis for $S \cap T$.

Soln $S \cap T = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \text{ with } \begin{matrix} x_1 - x_2 + x_3 - x_4 = 0 \\ x_1 + x_2 + x_3 + x_4 = 0 \end{matrix} \right\} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ -x_1 \\ -x_2 \end{bmatrix} \right\}$

because

$$\begin{aligned} & x_1 - x_2 + x_3 - x_4 = 0 \\ + \; & x_1 + x_2 + x_3 + x_4 = 0 \\ \hline & 2x_1 + 2x_3 = 0 \\ & x_3 = -x_1 \end{aligned} \qquad \left\| \qquad \begin{aligned} & x_1 - x_2 + x_3 - x_4 = 0 \\ - \; & (x_1 + x_2 + x_3 + x_4 = 0) \\ \hline & -2x_2 - 2x_4 = 0 \\ & x_4 = -x_2 \end{aligned} \right.$$

Standard Basis
$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right\}$

Remark:

To show $\text{Span}\{\underline{u}, \underline{v}\} = \text{Span}\{\underline{x}, \underline{y}, \underline{z}\}$

① $\underline{u}, \underline{v} \in \text{Span}\{\underline{x}, \underline{y}, \underline{z}\}$

② $\underline{x}, \underline{y}, \underline{z} \in \text{Span}\{\underline{u}, \underline{v}\}$

Two Constructive Thms:

- If $\{\underline{e}_1, \dots, \underline{e}_k\}$ Spans V then some subset is a basis
  $\longrightarrow$ Find linear comb. and throw them out.

- If $\{\underline{e}_1, \dots, \underline{e}_k\}$ is linearly indp. Then you can extend to a basis of V.
  $\longrightarrow$ Add elements in $V \setminus \text{span}\{\dots\}$

Ex. $\{1+m+m^2,\ 1-m,\ 1+m,\ 1+m^2,\ 1-m+m^2\}$

Find subset that is a basis

Plan: Find linear dependence relations

$\boxed{\text{lucky guess}} \rightarrow$ $(1+m+m^2) = (1+m) + (1+m^2) - \frac{1}{2}((1-m)+(1+m))$

$$= -\tfrac{1}{2}(1-m) + \tfrac{1}{2}(1+m) + 1(1+m^2)$$

Remove $(1+m+m^2)$

* To Find dependence relations :

Solve $\underline{0} = a\underline{v} + \dots$

$$0 = a(1-m) + b(1+m) + c(1+m^2) + d(1-m+m^2)$$
$$= (a+b+c+d) + (-a+b-d)m + (c+d)m^2$$

Look for solution not all 0.

$a+b+c+d=0$

$-a+b-d=0$ $\gg$ $2b+c=0 \longrightarrow b = -\frac{1}{2}c$

$c+d=0 \longrightarrow d=-c$ $\longrightarrow$ $a = \frac{1}{2}c$

if $c=2$ then $\begin{cases} a=1 \\ b=-1 \\ c=2 \\ d=-2 \end{cases}$

" Problem Solving "

ch 2. Ex 8. If V is a vector space then

(a) $a.v=0$ if & only if $a=0$ or $v=0$;

(b) $a.v=v$ if and only if $a=1$ or

Proof: (a) ($\Leftarrow$) $0.v=(1-1).v = 1.v+(-1)v = v-v=0$

$a.0 = a(0-0) = a.0 + a.(-0) = a.0 + a.(-1).0 = a.0 + (-a).0 = (a.0)-(a.0)=0$

($\Rightarrow$) If $a.v=0$ and $a \neq 0$ Same for a

$\frac{1}{a}.a.v = \frac{1}{a}.0$

$1.v = 0 \Rightarrow v=0$

(b) ($\Leftarrow$) $1.v=v$ (by axiom 8)

$a.0=0$ by above

($\Rightarrow$) $a.v=v$

$(a.v)-v = v-v$

$(a-1).v = 0 \Rightarrow v=0$ or $a-1=0$

$a=1$ by above.

Recall: To show $S=T$

① $S \subseteq T$

② $T \subseteq S$

To show (...) if and only if (...)

① ($\Rightarrow$)

② ($\Leftarrow$)

ch 2. Ex 13. Prove: If $v, w \in V$ then there is a unique $x \in V$ with $v+x=w$

Two parts. ① x exists.

② Only one x

Proof: (existance) Let $x = w-v$ $= v + (w-v) = v + (-v+w)$

$= (v+-v)+w$

$= 0 + w = w$

(Uniquness) Suppose y has $v+y=w$

$(v+y)-v = w-v$

$(y+v)-v = w-v \Rightarrow y+0 = w-v$ ∎ ✓

Ch3. Ex 8

Prove: $Fun(S,A) \cap Fun(S,B) = Fun(S, A \cup B)$

Proof: $(\subseteq)$ If $f \in Fun(S,A) \cap Fun(S,B)$ then $F \in Fun(S,A)$ so $f(s) = 0$ for $s \in A$

also $f \in Fun(S,B)$ so $f(s) = 0$ for $s \in B$

thus $f(s) = 0$ for $s \in A \cup B$ so $f \in Fun(S, A \cup B)$

$(\supseteq)$ Similar.

Ch3. Ex 9.

Prove. If $\phi : V \to \mathbb{R}$ with $\begin{cases} \phi(\underline{v}+\underline{w}) = \phi(v) + \phi(w) \\ \phi(r\underline{v}) = r\,\phi(v) \end{cases}$

Then $Nul(\varphi) = \{\underline{v} \text{ where } \varphi(\underline{v}) = 0\}$ is a Subspace.

Proof: ⓪ $\underline{0} \in V$ has $\varphi(\underline{0}) = \varphi(0 \cdot \underline{0}) = 0 \cdot \varphi(\underline{0}) = 0$

So $Nul(\varphi)$ is not empty.

① If $\underline{v}, \underline{w} \in Nul(\varphi)$ then $\varphi(v) = 0 = \varphi(\underline{w})$

So $\varphi(\underline{v}+\underline{w}) = \varphi(\underline{v}) + \varphi(\underline{w}) = 0 + 0 = 0$

thus $\underline{v}+\underline{w} \in Nul(\varphi)$

② Same ... ☒

Ch3. Ex 14. Prove: (1) + and · of Complex numbers is Commutative.

(2) $a \cdot (b+c) = ab + ac$ (with · is distributive)

(3) $a \cdot b \Rightarrow a = 0$ or $b = 0$

(4) $az + b = c$ has unique solution.

Proof: (1) ⊕ $(a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2) i$

$= (a_2 + a_1) + (b_2 + b_1) i$

$= (a_2 + b_2 i) + (a_1 + b_1 i)$

⊙

$(a_1 + b_1 i)(a_2 + b_2 i) = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1) i$

$= (a_2 a_1 - b_2 b_1) + (a_2 b_1 + a_1 b_2) i$

$= (a_2 + b_2 i)(a_1 + b_1 i)$

(3) $0 = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = (r_1 r_2) e^{i(\theta_1 + \theta_2)}$

So $r_1 r_2 = 0$ thus $r_1 = 0$ or $r_2 = 0$

Use Euler's Formula.

Ch3. Ex 15.

Prove: $l_0 = \{(a_0, a_1, a_2, \dots)\}$ is a vector space.

Proof: A1. $(a_0, a_1, \dots, a_i, \dots) + (b_0, b_1, \dots, b_i, \dots)$

$= (\dots, a_i + b_i, \dots)$

$= (\dots, b_i + a_i, \dots)$

$= (\dots, b_i, \dots) + (\dots, a_i, \dots)$

A2. Similar

A3. $\underline{0}$ is $(0, 0, 0, \dots)$

$(a_0, \dots, a_i, \dots) + (0, \dots, 0_i, \dots) = (a_0 + 0, \dots, a_i + 0, \dots) = (a_0, \dots, a_i, \dots)$ ✓

Ch3. Ex 16.

Proof: Claim l_∞ is a subspace of l_0

(0) $\underline{0} = (0, \dots)$ is bounded by 1 so $\underline{0} \in l_0$

(1) $(a_0, a_1, \dots)$ bounded by M then $(a_0, \dots) + (b_0, \dots) = (a_0 + b_0, \dots)$ bounded by $M+N$ (closed under +)

Ch4. Ex 29. Prove: If $\underline{v} + \underline{w} + \underline{x} = 0$ then $\mathrm{Span}(\underline{v}, \underline{w}) = \mathrm{span}\{\underline{w}, \underline{x}\}$

Proof:

($\subseteq$) $\underline{v} \in \mathrm{Span}\{\underline{w}, \underline{x}\}$ because $\underline{v} = (-1) \cdot w + (-1) \cdot \underline{x}$

$\underline{w} \in \mathrm{Span}\{\underline{w}, \underline{x}\}$ because $\underline{w} = 1 \cdot w$

thus if $y \in \mathrm{Span}\{\underline{v}, \underline{w}\}$, then $y \in \mathrm{Span}\{\underline{w}, \underline{x}\}$

$y = a\underline{v} + b\underline{w} = a((-1) w + (-1) \cdot x) + b \cdot w$

$= (b-a)\underline{w} + (-1) \cdot \underline{v}$

($\supseteq$) Same.

ch 4. Ex 16. Prove: If $E \subseteq F$ then $\mathrm{Span}(E) \subset \mathrm{Span}(F)$

Proof: Let $E \subseteq F$. If $\underline{y} \subseteq \mathrm{Span}(E)$ then $\underline{y} = a_1\underline{e}_1 + a_2\underline{e}_2 + \dots + a_n\underline{e}_n \quad \forall e_i \in E$

But $E \subseteq F$ so let $\underline{f}_i = \underline{e}_i \subseteq F$ and $b_i = e_i$

Then $\underline{y} = b_1\underline{f}_i + \dots + b_n\underline{f}_n$ for $\forall f_j \in F$

Thus $\underline{y} \in \mathrm{Span}(F)$ ✓

Prove: $\mathrm{Span}(E \cup F) = \mathrm{Span}(E) + \mathrm{Span}(F)$

$(\subseteq)$ If $\underline{v} \in \mathrm{Span}(E \cup F)$ then

$\underline{v} = a_1e_1 + \dots + a_n\underline{e}_n + b_1\underline{f}$

$\vdots$

ch 4. Ex 13. Prove $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\} = \mathrm{Span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\-1\\1\end{bmatrix}\right\}$

$\begin{cases} \begin{bmatrix}1\\1\\0\end{bmatrix} \in \mathrm{Span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\-1\\1\end{bmatrix}\right\} \longrightarrow \begin{bmatrix}1\\1\\0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\\1\end{bmatrix} + (-\frac{1}{2})\begin{bmatrix}-1\\-1\\1\end{bmatrix} \\ \begin{bmatrix}0\\0\\1\end{bmatrix} \in \mathrm{Span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\-1\\1\end{bmatrix}\right\} \longrightarrow \begin{bmatrix}0\\0\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}-1\\-1\\1\end{bmatrix} \checkmark \end{cases}$

If $\underline{v} \in \mathrm{Span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\-1\end{bmatrix}\right\}$ then $\underline{v} = a\begin{bmatrix}1\\1\\0\end{bmatrix} + b\begin{bmatrix}0\\0\\1\end{bmatrix}$

$= a\left(\frac{1}{2}\begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}-1\\-1\\1\end{bmatrix}\right) + b\left(\frac{1}{2}\begin{bmatrix}1\\1\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}-1\\-1\\1\end{bmatrix}\right)$

$\underline{v} = (\frac{1}{2}a + \frac{1}{2}b)\begin{bmatrix}1\\1\\1\end{bmatrix} + (-\frac{1}{2}a + \frac{1}{2}b)\begin{bmatrix}-1\\-1\\1\end{bmatrix}$

Then $\underline{v} \in \mathrm{Span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\-1\\1\end{bmatrix}\right\}$

ch4. 26. Is $\left\{\begin{bmatrix}x\\2m+1\end{bmatrix}\right\}$ a Subspace of $\mathbb{R}^2$?

No. $\begin{bmatrix}0\\0\end{bmatrix} \notin \left\{\begin{bmatrix}0\\2(0)+1\end{bmatrix}\right\}$

$\left\{\begin{bmatrix}x\\y\end{bmatrix} \text{ where } y - 2m = 0\right\}$

Ch5. Ex 18. Prove: $\{\underline{v}, \underline{w}, x\}$ independent $\Rightarrow \{\underline{v}+\underline{w}, \underline{w}+\underline{x}, \underline{v}+\underline{x}\}$ indep.

Proof. Suppose $a(\underline{v}+\underline{w}) + b(\underline{w}+\underline{x}) + c(\underline{v}+\underline{x}) = 0$

$(a+c)\cdot\underline{v} + (a+b)\underline{w} + (b+c)\cdot\underline{x} = \underline{0}$

$\{\underline{v}, \underline{w}, \underline{x}\}$ indep so

$\left.\begin{aligned} a+b&=0\\ a+c&=0\\ b+c&=0 \end{aligned}\right\} \quad \begin{aligned} a&=0\\ b&=0\\ c&=0 \end{aligned}$

So $\{\underline{v}+\underline{w}, \underline{w}+\underline{x}, \underline{v}+\underline{x}\}$ is linearly indep.

6.3 Using Bases

From 6.1 A Basis for V is B with

① B linearly indep.

② $\text{Span}(B) = V$

$\longrightarrow$ 6.3 Every $v \in V$ can be written uniquely as

$$v = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \qquad B = \{b_1, \ldots, b_n\}$$

$\updownarrow$ "Coordinates" of v relative to B

$$(a_1, a_2, \ldots, a_n) \longleftrightarrow \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n$$

Ex. Find coords of $2 - x = a(1-x) + b(1+x)$ in $P_1(\mathbb{R})$ relative to $\{(1-x), (1+x)\}$

$$2 - x = a(1-x) + b(1+x)$$

$$\downarrow$$

$$\begin{cases} 2 = a + b \\ -1 = -a + b \end{cases}$$

$1 = 2b \longrightarrow b = 1/2$ So $a = 3/2$ $\Rightarrow$ Answer $(3/2, 1/2)$

Ex. Find coords of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ relative to $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \underset{1}{a} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \underset{0}{b} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \underset{0}{c} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ Answer

$(1, 0, 0) \longleftrightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Ex. Let $S = \{1, 2, 3\}$

$\{f, g, h\}$ is basis for $\text{Fun}(S)$

f is $f(1) = 1$, $f(2) = 1$, $f(3) = 1$

g is $g(1) = 1$, $g(2) = 1$, $g(3) = 0$

h is $h(1) = 1$, $h(2) = 0$, $h(3) = 0$

Find coords of x_1, x_2, x_3 in this basis.

$x_1 = af + bg + ch \rightsquigarrow$ equivalent at Points:

$$\begin{bmatrix} x_1(1) = af(1) + bg(1) + ch(1) \\ x_1(2) = af(2) + bg(2) + ch(2) \\ x_1(3) = af(3) + bg(3) + ch(3) \end{bmatrix} \qquad \begin{bmatrix} 1 = a + b + c \\ 0 = a + b \\ 0 = a \end{bmatrix} \left. \vphantom{\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}} \right\} \begin{matrix} a = 0 \\ b = 0 \\ c = 1 \end{matrix}$$

$x_1 \longleftrightarrow (0, 0, 1) \quad x_2 \longleftrightarrow (0, 1, -1)$

Basis Extension Thm: If $\mathcal{U} \subset \mathcal{V}$ (subspace of vector sp) with E a basis for $\mathcal{U}$. You can Extend E to $\mathcal{B} \supseteq E$ a basis for $\mathcal{V}$.

Proof sketch:

If $\mathrm{Span}(E) = \mathcal{V}$ then let $\mathcal{B} = E$

otherwise Pick $\underline{v}_1 \in \mathcal{V} \setminus \mathrm{Span}(E)$

$E \cup \{\underline{v}_1\}$ is indep by construction.

(If dependent then one vector is in Span of Previous

$\rightarrow$ not a Vector in E because E is indep.

$\rightarrow$ not $\underline{v}_1$ because $\underline{v}_1 \notin \mathrm{Span}(E)$)

If $\mathrm{Span}(E \cup \{\underline{v}_1\}) = \mathcal{V}$ then let $\mathcal{B} = E \cup \{\underline{v}_1\}$

Otherwise repeat.

This must stop eventually as long as $\dim(V) < \infty$ ☒

Application:

- If $\{e_1, \dots, e_n\}$ linearly indep. elements of $\mathcal{V}$ with $\dim = n$, then $\{\underline{e}_1, \dots, \underline{e}_n\}$ is basis for $\mathcal{V}$.

Proof.

If $\mathrm{Span}\{\underline{e}_1, \dots, \underline{e}_n\} \neq V$ then extend to get a basis for $\mathcal{V}$. The basis would have $> n$ elements. But $\dim(V) = n$!

- If $\{\underline{e}_1, \dots, \underline{e}_n\}$ with $\mathrm{Span}\{\underline{e}_1, \dots, \underline{e}_n\} = \mathcal{V}$, $\dim(V) = n$ Then $\{\underline{e}_1, \dots, \underline{e}_n\}$ is a basis for $\mathcal{V}$.

Proof.

If not a basis we can Throw away dependent elements to get a basis with $< n$ elements But $\dim(V) = n$!

Exercise 17. If $\{\underline{v}_1, \dots, \underline{v}_n\} \subset \mathcal{V}$ with $\dim(V) = n$ Then $\mathrm{Span}\{\underline{v}_1, \dots, \underline{v}_n\} = \mathcal{V}$ if and only if $\{\underline{v}_1, \dots, \underline{v}_n\}$ linearly indep.

Exercise 10. If S, T subspaces of f.d. vector space then

$$\dim(S+T) = \dim(S) + \dim(T) - \dim(S \cap T)$$

Let E be a basis for S.

element in E $= \dim(S)$

Extend E to $E \cup F$ a basis for $(S+T)$

Shrink $E \cup F$ to G a basis for T

(Remove elements not in T)

$$\underbrace{\#(\text{elements of } E \cup F)}_{\dim(S+T)} = \underbrace{\#(\text{elements of } E)}_{\dim(S)} + \underbrace{\#(\text{elements of } E \cup F \text{ in } T)}_{\dim(T)} - \underbrace{\#(\text{elements of } E \text{ in } T)}_{\dim(S \cap T)}$$

If $\underline{e} \in E$ not T then $\underline{e}$ counted in #(elements E) but not in others.

$\underline{e} \in E$ and T then $\underline{e}$ counted by all three

$\underline{f} \in F$ then $\underline{f} \notin E$ and $\underline{f} \in T$ so

$\underline{f}$ is counted only by #(elements of $E \cup F$ in T)

Chapter 8. Linear Transformations

8.1 Definitions

Standard example: Linear Transformation $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

T

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \\ a_{31}x_1 + a_{32}x_2 \end{bmatrix}$$

$$\updownarrow$$

$$y_1 = a_{11}x_1 + a_{12}x_2$$
$$y_2 = a_{21}x_1 + a_{22}x_2$$
$$y_3 = a_{31}x_1 + a_{32}x_2$$

$$\updownarrow$$

Matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Idea: Linear Transf. "acts like matrix multiplication"

Def. A linear transformation $T: V \longrightarrow W$ between vector spaces.
is a map with $T(\underline{v}_1 + \underline{v}_2) = T(\underline{v}_1) + T(\underline{v}_2)$

$T(k \cdot \underline{v}) = k \cdot T(\underline{v})$

T "respects" vector space structure (Preserves)

Exercise 6: Let $T: \mathbb{R} \longrightarrow \mathbb{R}$ a linear transformation

Prove: There is a number a_T with $T(x) = a_T \cdot x$

Proof. Let $a_T = T(1)$

Then $T(x) = T(x \cdot 1) = x \cdot T(1) = x \cdot a_T$

$= a_T x$

Exercise 7. Let $T: \mathbb{R} \to \mathbb{R}$ a linear Transformation

If $T(3) = -4$ what is $T(7)$?

Sol. $T(3) = -4$
$\parallel$
$T(3 \cdot 1)$
$\parallel$
$3 \cdot T(1)$

$a_T = T(1) = -\frac{4}{3}$

$T(7) = \frac{-4}{3} \cdot 7$

Exercise 8. Let $T: \mathbb{R}^2 \to \mathbb{R}$ a linear Transf.

Prove: There is $A_T = [a_T \; b_T]$ with $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = a_T x + b_T y$

Proof. Let $a_T = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$, $b_T = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ } standard basis elements

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$
$$= x\,T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + y\,T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$
$$= a_T x + b_T y$$

Big Fact: You can do this for all $T: \mathbb{R}^n \to \mathbb{R}^m$

Ex. $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$

$$R_\theta\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix} \longleftarrow \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

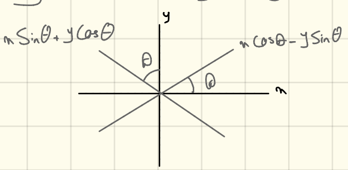
Then R_θ is linear transf.

$$R_\theta\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = R_\theta\left(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}\right) = \left[\begin{pmatrix} (x_1+x_2)\cos\theta - (y_1+y_2)\sin\theta \\ (x_1+x_2)\sin\theta + (y_1+y_2)\cos\theta \end{pmatrix}\right]$$

$$= \begin{bmatrix} x_1\cos\theta - y_1\sin\theta \\ x_1\sin\theta + y_1\cos\theta \end{bmatrix} + \begin{bmatrix} x_2\cos\theta - y_2\cos\theta \\ x_2\sin\theta + y_2\sin\theta \end{bmatrix} = R_\theta\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + R_\theta\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

Also $R_\theta\left(c\begin{bmatrix} x \\ y \end{bmatrix}\right) = \dots = c R_\theta\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$

By the way R_θ rotates by θ degrees.



Ex. $D: P(\mathbb{R}) \to P(\mathbb{R})$

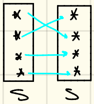
$$D(p) = \frac{d^3}{dx^3}p + 3\frac{d^2}{dx^2}p + x\frac{d}{dx}p + 2p \quad \text{(linear differential operator)}$$

Linear Transf. b\c

$$\frac{d}{dx}(p+q) = \frac{d}{dx}p + \frac{d}{dx}q$$
$$x\cdot(p+q) = xp + xq$$
} → Apply many times.

⟹ Derivatives are linear transf.!

Ex. Recall a permutation a set is a bijection $\alpha : S \to S$

Permutation: 

Ex. $S = \{1,2,3\}$

$\left.\begin{array}{l}\alpha(1)=2\\ \alpha(2)=1\\ \alpha(3)=3\end{array}\right\}$ is a permutation.

$\left.\begin{array}{l}\alpha(1)=2\\ \alpha(2)=1\\ \alpha(3)=2\end{array}\right\}$ is not!

Every permutation $\alpha : S \to S$ "induces" a linear transf. on $Fun(S)$:

$$\alpha^* : Fun(S) \longrightarrow Fun(S)$$

$$(\alpha^*(f))(s) = f(\alpha(s))$$ "change of variables"

Ex. $S = \{x, y, z\}$ $\begin{cases}\alpha(x) = y\\ \alpha(y) = x\\ \alpha(z) = z\end{cases}$

If $f = 2\chi_x + 3\chi_y - \chi_z$

$\alpha^* f = 3\chi_x + 2\chi_y - \chi_z$

Review, Done So far

Def: $T : V \to W$ is a linear transformation if

$T(v_1 + v_2) = T(v_1) + T(v_2)$ and $T(k \cdot v) = kT(v)$

Ex. $T : \mathbb{R} \to \mathbb{R}$

$T(x) = a_T \cdot x$

Ex. $T : \mathbb{R}^2 \to \mathbb{R}$

$T\left(\begin{bmatrix} x \\ y\end{bmatrix}\right) = a_T \cdot x + b_T \cdot y$

Ex. $T\left(\begin{bmatrix} x \\ y\end{bmatrix}\right) = \begin{bmatrix} T_1\left(\begin{bmatrix} x \\ y\end{bmatrix}\right) \\ T_2\left(\begin{bmatrix} x \\ y\end{bmatrix}\right)\end{bmatrix}$

$= \begin{bmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix} x \\ y\end{bmatrix} = A_T\begin{bmatrix} x \\ y\end{bmatrix}$

Exercises:

① Linear Transf. / not linear Transf.?

(a) $T\left(\begin{bmatrix} x \\ y\end{bmatrix}\right) = \begin{bmatrix} 2x - y \\ y\end{bmatrix}$ Yes

NOTE: Linear Transf. take linear Comb. of input to get output.

(b) $T\left(\begin{bmatrix} x \\ y \\ z\end{bmatrix}\right) = \begin{bmatrix} z \\ x+y\end{bmatrix}$ Yes.

(a) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2 \\ y^2 \end{bmatrix}$ NO

(b) $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y+z \\ 1 \end{bmatrix}$ NO

(c) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ NO $\rightarrow \begin{cases} T\left(2\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ 2\,T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = 2\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} \end{cases}$

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ Yes. $\rightarrow \begin{cases} T\left(2\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ 2\,T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = 2 \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{cases}$

(d) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} xy \\ x+y \end{bmatrix}$ NO

Exercise 10: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ show there are numbers a, b so that

$$(T \cdot T + aT + b\,Id) = 0 : \mathbb{R}^2 \to \mathbb{R}^2$$

Ex. $V = \mathbb{R}$

W = exponential vector space $\quad a+b = ab \quad ka = a^k \quad \left(\begin{array}{l} \underline{0} = 1 \\ -a = 1/a \end{array} \right)$

$T: V \to W$

$T(\underline{v}) = e^v$

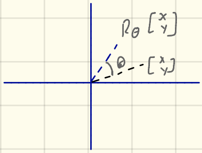
$$T(\underline{v}+\underline{w}) = e^{v+w} = \underline{e^v \cdot e^w}$$
$$= \underline{e}^v + \underline{e}^w$$
$$= T(\underline{v}) + T(\underline{w})$$

$$T(k\underline{v}) = e^{kv} = (e^v)^w = k \cdot e^v = k\,T(v)$$

Basic Examples

8.1

$R_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ rotating by θ



8.2

$P : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ Projection onto xy-plane

$$P\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

$S : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ reflection across $x+y+z=0$ plane.

$$S\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{3}\begin{bmatrix} x-2y-2z \\ -2x+y-2z \\ -2x-2y+z \end{bmatrix}$$

$T : \mathbb{R}^3 \longrightarrow P_2(\mathbb{R})$

$$T\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a + bx + cx^2$$

8.1

$\frac{d}{dx} : P_n(\mathbb{R}) \longrightarrow P_n(\mathbb{R})$

$$\frac{d}{dx}(a_0 + a_1 x + \cdots + a_n x^n) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1}$$

8.2

Evaluation at s $\quad ev_s : Fun(S) \longrightarrow \mathbb{R}$

$$ev_s(f) = f(s)$$

If $S = \{s_1, s_2, \ldots, s_n\}$

$$ev_S(f) = \begin{bmatrix} f(s_1) \\ f(s_2) \\ \vdots \\ f(s_n) \end{bmatrix}$$ is linear transf.

$$Fun(S) \longrightarrow \mathbb{R}^n$$

8.3 Properties of Linear Transformations

Linear Transformations preserve all vector space structure. ($T: V \longrightarrow W$ Linear Transf)

Prop: $T(\underline{0}_V) = \underline{0}_W$ (Preserve $\underline{0}$ vector)

Proof: $T(\underline{0}_V) = T(0 \cdot \underline{0}_V) = 0 \cdot T(\underline{0}_V)$
$= \underline{0}_W$

$T(-v) = -T(\underline{v})$ (Preserve negatives)

Prop: $T(a_1\underline{v}_1 + \cdots + a_n\underline{v}_n) = a_1 T(v_1) + \cdots + a_n T(\underline{v}_n)$

Proof: $T(a_1 v_1 + \cdots + a_n v_n) = T(a_1\underline{v}_1) + \cdots + T(a_n\underline{v}_n)$
$= a_1 T(\underline{v}_1) + \cdots + a_n T(\underline{v}_n)$

Prop: If $\mathcal{U} \subset V$ is a subspace with
So is $T(\mathcal{U}) \subset W$

Proof: . $\underline{0} \subset \mathcal{U}$ so $\underline{0} = T(\underline{0}) \in T(\mathcal{U})$. ($T(\mathcal{U}) \neq \emptyset$)

. If $\underline{w}_1, \underline{w}_2 \in T(\mathcal{U})$ then $\underline{w}_1 = T(v_1)$ same $\underline{v}_1, \underline{v}_2 \in \mathcal{U}$
$\underline{w}_2 = T(v_2)$

$\underline{w}_1 + \underline{w}_2 = T(\underline{v}_1) + T(\underline{v}_2) = T(v_1 + v_2) \in T(\mathcal{U})$

. Same for scaling

NOTE. Linear transformations may not preserve independence.

$\rightarrow T: \mathbb{R} \rightarrow \{0\}$ $\quad \{1\} \subset \mathbb{R}$ is linearly independent but $\{T(1)\} = \{\underline{0}\}$ is not.

$\rightarrow P: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ Project to x-axis

$P\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$

$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \subset \mathbb{R}^2$

$\left\{ T\begin{bmatrix} 1 \\ 0 \end{bmatrix}, T\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ not linear. Indep.

Ex If $T(E)$ is indep. then E is indep.

Prop $T(\text{Span}(E)) = \text{Span}(T(E))$

$T(\text{Span}\{\underline{e}_1, \dots, \underline{e}_n\}) = \text{Span}\{T(\underline{e}_1), \dots, T(\underline{e}_n)\}$

Proof

If $\underline{w} \in T(\text{Span}(E))$ then $\underline{w} = T(\underline{v})$ where $\underline{v} \in \text{Span}(E)$

$\underline{v} \in \text{span}(E)$ so $\underline{v} = a_1\underline{e}_1 + \dots + a_n \underline{e}_n$ with $\underline{e}_i \in E$

Thus

$\underline{w} \in T(a_1\underline{e}_1 + \dots + a_n \underline{e}_n) = a_1 T(\underline{e}_1) + \dots + a_n T(\underline{e}_n) \in \text{Span}(T(E))$

If $\underline{w} \in \text{Span}(T(E))$

... Same thing but backwards ...

Prop: $\text{Ker}(T) = T^{-1}(\{0\}) = \{\underline{v} \in V \text{ with } T(\underline{v}) = \underline{0} \in W\}$ is a subspace of V.
(In another class this would be called the "nullspace")

Proof:
- $T(\underline{0}) = \underline{0}$ so $\underline{0} \in \ker(T)$ (nonempty)
- $T(\underline{v}_1 + \underline{v}_2) = T(\underline{v}_1) + T(\underline{v}_2)$ so

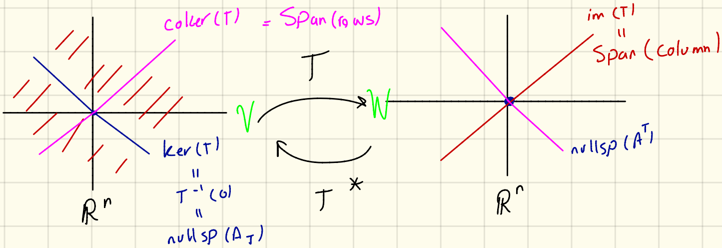
  if $\underline{v}_1, \underline{v}_2 \in \text{Ker}(T)$ then $T(\underline{v}_1 + \underline{v}_2) = 0 + 0 = 0$
- Same for scaling.

By the way if $X \subset W$ a subspace $T: V \to W$
then $T^{-1}(X) \subset V$ a subspace $T^{-1}(X) \to X$

NOTE. $T^{-1}(T(U))$ may not be U

$T^{-1}(T(U)) \supset U$

Linear Transf. do something interesting to subspaces.



Book Strang Intro. to Linear Algebra.

Prop. The Composition of linear transformations is a linear transf.

(i.e. $S: V_1 \to V_2$ and $T: V_2 \to V_3$ linear Transf. Then $T \circ S: V_1 \to V_2$ is linear Transf.)

Proof.

$$(T \circ S)(\underline{v}_1 + \underline{v}_2) = T(S(\underline{v}_1 + \underline{v}_2))$$
$$= T(S(\underline{v}_1) + S(\underline{v}_2))$$
$$= T(S(v_1)) + T(S(v_2))$$
$$= (T \circ S)(v_1) + (T \circ S)(v_2)$$

Scaling is the Same.

Goal: Keep track of dimensions of subspaces.

$$\mathcal{U} \longrightarrow T(\mathcal{U})$$
$$T^{-1}(\mathcal{X}) \longrightarrow \mathcal{X}$$

8.4 Images and Kernels (dimension formula "Rank thm")

Def $\ker(T) = T^{-1}(\{0\}) \longrightarrow$ stuff that T forget

- $\operatorname{im}(T) = T(V) = \{\underline{w} \in W \text{ with } \underline{w} = T(\underline{v}) \text{ for some } \underline{v}\}$

Prop: $\ker(T)$ and $\operatorname{im}(T)$ are subspaces.

Ex. $R_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ rotation

$\ker(R_\theta) = \left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$ $\qquad \begin{bmatrix}0\\0\end{bmatrix} = R_\theta\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x\cos\theta - y\sin\theta\\ x\sin\theta + y\cos\theta\end{bmatrix}$

$\operatorname{im}(R_\theta) = \mathbb{R}^2$ $\qquad \operatorname{im}(R_\theta) = R_\theta\left(\operatorname{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}\right)$

$$= \operatorname{Span}\left(\left\{R_\theta\begin{bmatrix}1\\0\end{bmatrix}, R_\theta\begin{bmatrix}0\\1\end{bmatrix}\right\}\right)$$
$$= \operatorname{Span}\left(\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}, \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}\right)$$

$\left\{\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}, \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}\right\}$ linearly indep.

2 vectors in $\mathbb{R}^2$ (dim = 2 space)
So Span = $\mathbb{R}^2$

Ex: $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x-y \\ x \end{bmatrix}$

$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$\ker(T) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

$\text{im}(T) = T\left(\text{Span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)\right)$

$= \text{Span}\left(T\begin{bmatrix} 1 \\ 0 \end{bmatrix}, T\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

$= \text{Span}\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right)$

these are indep. so they are a basis for im(T)

$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ w/ $x + y - z = 0 \}$

Ex. $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x + y + z$

$T: \mathbb{R}^3 \longrightarrow \mathbb{R}^1$

$\ker(T) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ with } x + y + z = 0 \right\}$

$= \left\{ \begin{bmatrix} x \\ y \\ -x-y \end{bmatrix} \right\}$

$= \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$

$\text{im}(T) = \mathbb{R}^1$

$T\left(\begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}\right) = x$

Thm: $\dim(\ker(T)) + \dim(\text{im}(T)) = \dim(V)$ "Dimension Formula"

(amount of info T forgets) + (amount of info T sees) = amount of information going into T

Proof: Let $\{k_1, \dots, k_s\}$ be a basis for $\ker(T)$ (so $T(k_i) = 0$)

Expand to:

$\{k_1, \dots, k_s, e_1, \dots, e_t\}$ a basis for V.

(Goal: $\{T(e_1), \dots, T(e_t)\}$ is basis for $\operatorname{im}(T)$)

$\operatorname{Span}\{T(e_1), \dots, T(e_n)\} = \operatorname{im}(T)$ b/c

$\operatorname{im}(T) = \operatorname{Span}\{\cancelto{0}{T(k_1)}, \dots, \cancelto{0}{T(k_s)}, T(e_1), \dots, T(e_t)\}$

$= \operatorname{Span}\{T(e_1), \dots, T(e_n)\}$

Suppose $0 = a_1 T(e_1) + \dots + a_t T(e_t)$

$= T(a_1 e_1 + \dots + a_t e_t)$

So $a_1 e_1 + \dots + a_t e_t \in \ker(T) = \operatorname{Span}\{k_1, \dots, k_s\}$

But $\operatorname{Span}\{e_1, \dots, e_t\} \cap \operatorname{Span}\{k_1, \dots, k_s\} = \{0\}$

Thus $a_1 e_1 + \dots + a_t e_t = 0$

e_i indep. so $a_i = 0$

Thus $T(e_i)$ indep.

$(s) + (t) = (s+t)$ ∎

$T: \begin{pmatrix}\text{Vector space}\\ \text{structure in}\\ V\end{pmatrix} \longrightarrow \begin{pmatrix}\text{Vector space}\\ \text{structure in}\\ W\end{pmatrix}$

Subspace: $\begin{cases} \bullet \ker(T) = T^{-1}\{\underline{0}\} = \{\underline{v} \in V \text{ with } T(\underline{v}) = \underline{0}\} \\ \bullet \operatorname{im}(T) = T(V) = \{\underline{w} \in W \text{ with } \underline{w} = T(v) \text{ some } v\} \end{cases}$

NOTE: Use that $T(V)$ is a subspace to spot fake linear transf.

→ Ex: $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x^2 \\ y^2 \end{bmatrix}$ ← not a subspace so this is not lin. transf.

8.5 Fundamental Constructions (Linear Extensions)

Setup: $V \xrightarrow{T} W$

basis for V: $\left\{ \underline{b}_1, \underline{b}_2, \ldots, \underline{b}_n \right\}$ $\longrightarrow$ $\left\{ \underline{w}_1 = t(\underline{b}_1),\ \underline{w}_2 = t(\underline{b}_2),\ \ldots,\ \underline{w}_n \right\}$ Some Vectors in W

Thm: Given a basis $\{\underline{b}_1, \ldots, \underline{b}_n\}$ of V and a map $\{\underline{b}_1, \ldots, \underline{b}_n\} \xrightarrow{t} W$

There is a Unique linear transformation

$T : V \longrightarrow W$ with $T(\underline{b}_i) = t(\underline{b}_i)$

(The "linear extension" of t)

Proof If $\underline{v} \in V$ then $\underline{v} = a_1 \underline{b}_1 + \cdots + a_n \underline{b}_n$

Define
$$\begin{aligned} T(\underline{v}) &= T(a_1 \underline{b}_1 + \cdots + a_n \underline{b}_n) \\ &= a_1 T(\underline{b}_1) + \cdots + a_n T(\underline{b}_n) \\ &= a_1 t(\underline{b}_1) + \cdots + a_n t(\underline{b}_n) \\ &= a_1 \underline{w}_1 + \cdots + a_n \underline{w}_n \end{aligned}$$

Must show that this is a linear transformation.

If $\underline{v}_1, \underline{v}_2 \in V$ $\begin{cases} \underline{v}_1 = a_1 \underline{b}_1 + \cdots + a_n \underline{b}_n \\ \underline{v}_2 = \alpha_1 \underline{b}_1 + \cdots + \alpha_n \underline{b}_n \end{cases}$

$$\begin{aligned} T(\underline{v}_1 + \underline{v}_2) &= T((a_1 + \alpha_1)\underline{b}_1 + \cdots + (a_n + \alpha_n)\underline{b}_n) = (a_1 + \alpha_1) T(\underline{b}_1) + \cdots + (a_n + \alpha_n) T(\underline{b}_n) \\ &= (a_1 T(\underline{b}_1) + \cdots + a_n T(\underline{b}_n)) + (\alpha_1 T(\underline{b}_1) + \cdots + \alpha_n T(\underline{b}_n)) = T(\underline{v}_1) + T(\underline{v}_2) \end{aligned}$$

$T(k\underline{v}) = \cdots = kT(\underline{v})$
Same for scaling

Ex. Compute R_θ

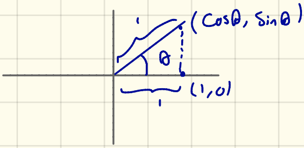
$\rightarrow$ figure out $R_\theta \begin{bmatrix}1\\0\end{bmatrix}$

$R_\theta \begin{bmatrix}0\\1\end{bmatrix}$

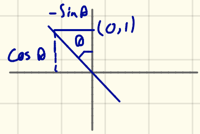
$\rightarrow$ Linear Extension

$R_\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$R_\theta \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$$



$$R_\theta \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$$



$$\Rightarrow R_\theta \begin{bmatrix}x\\y\end{bmatrix} = R_\theta \left(x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix} \right)$$

$$= x R_\theta \begin{bmatrix}1\\0\end{bmatrix} + y R_\theta \begin{bmatrix}0\\1\end{bmatrix}$$

$$= x \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} + y \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$$

$$= \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$$

Ex. Compute the linear extension of

$$T\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$$

$$T\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}2\\4\end{bmatrix}$$

Note: $\left\{ \begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}3\\2\end{bmatrix} \text{ is a basis of } \mathbb{R}^2 \right\}$

$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

① $\rightarrow$ Scratch work: $\Rightarrow \begin{bmatrix}x\\y\end{bmatrix} = a\begin{bmatrix}2\\3\end{bmatrix} + b\begin{bmatrix}3\\2\end{bmatrix}$

$\Rightarrow x = 2a + 3b$

$y = 3a + 2b$

$a = -\frac{1}{5}(2x - 3y)$

$b = -\frac{1}{5}(-3x + 2y)$

② $\Rightarrow$ $$T\begin{bmatrix}x\\y\end{bmatrix} = T\left(-\tfrac{1}{5}(2x-3y)\begin{bmatrix}2\\3\end{bmatrix} - \tfrac{1}{5}(-3x+2y)\begin{bmatrix}3\\2\end{bmatrix} \right)$$

$$= -\frac{1}{5}(2x-3y)\,T\begin{bmatrix}2\\3\end{bmatrix} - \frac{1}{5}(-3x+2y)\,T\begin{bmatrix}3\\2\end{bmatrix}$$

$$= -\frac{1}{5}(2x-3y)\begin{bmatrix}1\\2\end{bmatrix} - \frac{1}{5}(-3x+2y)\begin{bmatrix}2\\4\end{bmatrix}$$

$$= \begin{bmatrix} x - \frac{1}{5}y \\ 2x - \frac{2}{5}y \end{bmatrix}$$

Review

Prop: T is a linear extension with $T(\underline{b}_i) = \underline{w}_i$

$$\operatorname{Ker}(T) = 0 \iff \{\underline{w}_1, \dots, \underline{w}_n\} \text{ lin. Indep.}$$

Prop: T is a linear extension with $T(\underline{b}_i) = \underline{w}_i$

$$\operatorname{im}(T) = W \iff \operatorname{Span}\{\underline{w}_1, \dots, \underline{w}_n\} = W$$

→ **Cor:** If $\ker(T) = 0$ and $\operatorname{im}(T) = W$ then

$\{T(\underline{b}_1), \dots, T(\underline{b}_n)\}$ is a basis for W

Operation on Linear Transf.

Def. If $S, T : V \longrightarrow W$ then

$$(S+T)(\underline{v}) = S(\underline{v}) + T(\underline{v})$$

Prop.

$S+T$ is a linear Transformation.

Proof.

$$\begin{aligned}
(S+T)(\underline{v}_1 + \underline{v}_2) &= S(\underline{v}_1 + \underline{v}_2) + T(\underline{v}_1 + \underline{v}_2) \\
&= S(\underline{v}_1) + S(\underline{v}_2) + T(\underline{v}_1) + T(\underline{v}_2) \\
&= \big(S(\underline{v}_1) + T(\underline{v}_1)\big) + \big(S(\underline{v}_2) + T(\underline{v}_2)\big) \\
&= (S+T)(\underline{v}_1) + (S+T)(\underline{v}_2)
\end{aligned}$$

$$(S+T)(k \cdot \underline{v}) = \dots = k\,(S+T)(\underline{v}) \quad \blacksquare$$

Prop. $k \cdot T$ is a linear transformation.

Proof: $(k\cdot T)(\underline{v}_1+\underline{v}_2) = k \cdot T(\underline{v}_1+\underline{v}_2) = k.(T(\underline{v}_1)+T(\underline{v}_2))$
$= k\cdot T(\underline{v}_1) + k\cdot T(\underline{v}_2)$
$= (k\cdot T)(\underline{v}_1) + (k\cdot T)(\underline{v}_2)$
$\vdots$
etc $\blacksquare$

Def. $0: V \longrightarrow W$ the "Zero transformation" is $0(\underline{v}) = \underline{0}_W$

Def. $\mathcal{L}(V,W) = \{$ linear transformations $V \longrightarrow W \}$

Prop. $\mathcal{L}(V,W)$ is a Vector Space!

Most useful Example: $\mathcal{L}(V,\mathbb{R}) = V^*$ "dual Space of V"

Ex. $\mathcal{L}(\mathbb{R},\mathbb{R}) = \mathbb{R}^* = \{a^*\}$ where $a^*(x) = a.x$

Ex. $\mathcal{L}(\mathbb{R}^2,\mathbb{R}) = (\mathbb{R}^2)^* = \left\{\begin{bmatrix} a \\ b \end{bmatrix}^*\right\}$ where $\begin{bmatrix} a \\ b \end{bmatrix}^*\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = ax + by$

Ex. $\mathcal{L}(P_2,\mathbb{R}) = (P_2)^* = \{(\alpha+\beta+\gamma x^2)^*\}$ where $(\alpha+\beta x+\gamma x^2)^*(a+bx+cx^2)$
$= \alpha.a + \beta.b + \gamma.c$

Proof as "H.W"

8.6 Isomorphisms

Ex. $Fun(\{1,2,3\}) \xrightarrow{\cong} \mathbb{R}^3$

By $T(\chi_1) = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $T(\chi_2) = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $T(\chi_3) = \begin{bmatrix}0\\0\\1\end{bmatrix}$ $\Big\}$

$im(T) = \mathbb{R}^3$
$ker(T) = \{0\}$

Ex. $T: P_2 \xrightarrow{\cong} \mathbb{R}^3$

T takes <u>bases</u> to <u>bases</u>

By $T(1) = \begin{bmatrix}1\\0\\0\end{bmatrix}$
$T(2) = \begin{bmatrix}0\\1\\0\end{bmatrix}$
$T(3) = \begin{bmatrix}0\\0\\1\end{bmatrix}$

Def. $T: V \to W$ is a isomorphisms

if there is a linear transf. $S: W \to V$ $(S = T^{-1})$

with $(S \circ T)(v) = \underline{v}$ $\forall \underline{v} \in V$
$(T \circ S)(w) = \underline{w}$ $\forall \underline{w} \in W$

$$\begin{tikzcd} V \arrow[r, bend left, "T"] & W \arrow[l, bend left, "S"] \end{tikzcd}$$

$(S = T^{-1})$

Notation when there is an isomorphism $T: V \to W$
we say V and W are "isomorphic"
and write $\underline{V \cong W}$ $T: V \xrightarrow{\cong} W$

Ex. $T: Fun(\{1,2,3\}) \to \mathbb{R}^3$
$T(\chi_1) = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $T(\chi_2) = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $T(\chi_3) = \begin{bmatrix}0\\0\\1\end{bmatrix}$
$S: \mathbb{R}^3 \to Fun(\{1,2,3\})$

$S\begin{bmatrix}a\\b\\c\end{bmatrix} = a\chi_1(t) + b\chi_2(t) + c\chi_3(t) \stackrel{?}{=} f(m)$

$(S \circ T)(f)(m)$

$= (S \circ T)(f(1)\,\chi_1(m) + f_2\,\chi_2(m) + f(3)\,\chi_3(m))$

$= S\left(f(1)\begin{bmatrix}1\\0\\0\end{bmatrix} + f(2)\begin{bmatrix}0\\1\\0\end{bmatrix} + f(3)\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$

$= S\left(\begin{bmatrix}f(1)\\f(2)\\f(3)\end{bmatrix}\right) = f(1)\cdot\chi_1(m) + f(2)\cdot\chi_2(m) + f(3)\,\chi_3(m)$

$= f(m).$

$(T \circ S)\left(\begin{bmatrix}a\\b\\c\end{bmatrix}\right) = T\left(a\chi_1(m) + b\chi_2(m) + c\chi_3(m)\right)$

$= a\cdot\begin{bmatrix}1\\0\\0\end{bmatrix} + b\cdot\begin{bmatrix}0\\1\\0\end{bmatrix} + c\cdot\begin{bmatrix}0\\0\\1\end{bmatrix}$

$= \begin{bmatrix}a\\b\\c\end{bmatrix} \quad \blacksquare$

<u>Ex.</u> $T : P_2 \xrightarrow{\cong} \mathbb{R}^3$

$T(1) = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad T(m) = \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad T(x_2) = \begin{bmatrix}0\\0\\1\end{bmatrix}$

$S : \mathbb{R}^3 \longrightarrow P_2$

$S\begin{bmatrix}a\\b\\c\end{bmatrix} = a + bx + cx^2$

<u>Ex.</u> $T : P_2 \xrightarrow{\cong} \mathbb{R}^3$

$T(1) = \begin{bmatrix}1\\1\\1\end{bmatrix}, \quad T(x) = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad T(x^2) = \begin{bmatrix}1\\0\\0\end{bmatrix}$

$S : \mathbb{R}^3 \rightarrow P_2$

$S\begin{bmatrix}a\\b\\c\end{bmatrix} = c + (b-c)\,x + (a-b)\,x^2$

$T \circ S\begin{bmatrix}a\\b\\c\end{bmatrix}$

$= T\left(c + (b-c) + (a-b)\,x^2\right)$

$= c\begin{bmatrix}1\\1\\1\end{bmatrix} + (b-c)\begin{bmatrix}1\\1\\0\end{bmatrix} + (a-b)\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix}$

Goal: Equivalent definitions for $T: V \xrightarrow{\cong} W$

(a) T has an inverse $S: W \to V$

(b) $im(T) = W$
$Ker(T) = \{\underline{0}\}$ Known

(c) $T(\text{basis for } V)$ is a basis for W

Prop. If $Ker(T) = \{\underline{0}\}$ then

$\underline{w} \in im(T)$ has exactly one way of being written as

$$\underline{w} = T(\underline{v})$$

Proof: (looks like Proof that linear comb. of linearly indep. set are Unique.)

Let $\underline{w} \in im(T)$ and Suppose

$$T(v_1) = \underline{w}$$
$$T(v_2) = \underline{w}$$

Then
$$\begin{aligned}\underline{0} &= \underline{w} - \underline{w} \\ &= T(\underline{v}_1) - T(\underline{v}_2) \\ &= T(\underline{v}_1 - \underline{v}_2)\end{aligned}$$

$\longrightarrow$ So $\underline{v}_1 - \underline{v}_2 \in Ker(T)$. Thus $\underline{v}_1 - \underline{v}_2 = \underline{0}$

$\underline{v}_1 = \underline{v}_2$ ▨

Thm. T has an inverse if and only if $im(T) = W$
$ker(T) = \{\underline{0}\}$

Proof:

$(\Rightarrow)$ Suppose T has an inverse $S: W \longrightarrow V$

- if $\underline{w} \in W$ then $\underline{w} = (T \circ S)(\underline{w})$
$= T(S(\underline{w}))$

So $im(T) = W$

- if $\underline{v} \in V$ with $T(\underline{v}) = 0$
$\underline{v} = S \circ T(\underline{v}) = S(\underline{0}) = \underline{0}$

thus $Ker(T) = \{0\}$

$(\Leftarrow)$ Suppose $im(T) = W$ and $Ker(T) = \{0\}$

By Prop for every $\underline{w} \in im(T) = W$ there is exactly one $\underline{v}$ with $T(\underline{v}) = \underline{w}$

Let $S(\underline{w}) = \underline{v}$

Note:

$S \circ T(\underline{v}) = S(\underline{w}) = \underline{v}$ by construction

$T \circ S(\underline{w}) = T(\underline{v}) = \underline{w}$ by construction

Only need to show S is linear transf.

$$S(\underline{w}_1 + \underline{w}_2) = \underline{v}_1 + \underline{v}_2 \text{ where } \left.\begin{array}{l} T(\underline{v}_1) = \underline{w}_1 \\ T(\underline{v}_2) = \underline{w}_2 \end{array}\right\} T(\underline{v}_1 + \underline{v}_2) = \underline{w}_1 + \underline{w}_2$$

$$= S(\underline{w}_1) + S(\underline{w}_2)$$

$S(k \cdot \underline{w}) = \ldots = k\, S(\underline{w})$ $\square$

Ex: Is $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ an isomorphism?

$$T = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

$$T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$T\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

output is not a basis $\Rightarrow$ not an isomorphism

method #2

$$\operatorname{Ker}(T) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ with } \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} \right\}$$

$$= \operatorname{Span}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \neq \{0\}$$

$$\operatorname{im}(T) = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \right\} \neq \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right\} = \mathbb{R}^3$$

Ex: Is $\mathbb{R}^3 \longrightarrow \mathbb{R}^4$ an isomorphism?

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ x+y \\ x+y+z \\ x+y+z+w \end{bmatrix}$$

$$\underbrace{T\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad T\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad T\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad T\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{A basis for } \mathbb{R}^4}$$

A basis for $\mathbb{R}^4$ $\Rightarrow$ is an isomor.

NOTE.
- Computing $\operatorname{Ker}(T) \Longleftrightarrow$ checking independence of $T(\underline{b}_i)$
- Computing $\operatorname{im}(T) \Longleftrightarrow$ checking $\operatorname{Span}(\{T(\underline{b}_i)\}_i) = \mathbb{R}^n$

Lemma. If $\dim(V) = \dim(W)$ then
- $\ker(T) = \{\underline{0}\} \Longrightarrow T$ an isomorphism
- $\operatorname{im}(T) = W \Longrightarrow T$ an isomorphism

Thm. If V, W finite dim. then

$$\dim(V) = \dim(W) \iff V \cong W$$

Cor. $\dim(V) = n \iff V \cong \mathbb{R}^n$

$\longrightarrow$ from now on, we can focus on $\mathbb{R}^n$

Review of Chapter 8

Linear Transformations $T: V \longrightarrow W$ with $\begin{cases} T(\underline{v}_1 + \underline{v}_2) = T(v_1) + T(v_2) \\ T(k \cdot \underline{v}) = k\,T(v) \end{cases}$

Special Subspaces $\left\{ \begin{array}{l} T^{-1}\{\underline{0}\} = \operatorname{Ker}(T) \longrightarrow \{0\} \\ V \longrightarrow \operatorname{im}(T) = T(V) \end{array} \right\}$

- A vector space is $\operatorname{span}\{\underline{b}_1, \dots, \underline{b}_n\}$
- A linear transf. is linear extension of $\{b_1, b_2, \dots, b_n\}$ basis of V $\quad b_1 \rightarrow T(\underline{b}_1)$, $b_2 \rightarrow T(\underline{b}_2)$, $b_3 \rightarrow T(\underline{b}_3)$

If $\ker(T) = \{0\}$ then $T(\text{indep set in } V)$ is indep set in W.

If $\operatorname{im}(T) = W$ then $T(\text{spanning set in } V)$ spans W.

$\longrightarrow$ $\begin{array}{l} \operatorname{Ker}(T) = \{0\} \\ \operatorname{im}(T) = W \end{array} \iff T(\text{basis for } V)$ is basis for W.

$\Rightarrow$ "T is an isomorphism. $V \cong W$" $\iff T$ has "inverse" S

Combinations of linear transformations

$$(S+T)(\underline{v}) = S(\underline{v}) + T(\underline{v})$$

$$(k.T)(\underline{v}) = k.T(\underline{v})$$

$$(S \circ T)(\underline{v}) = S(T(\underline{v})).$$

Chapter 9 . Examples

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y+z \\ 0 \\ 0 \end{bmatrix} \quad , \quad S\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ x \\ y \end{bmatrix}$$

$$\mathrm{im}(T) = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$$

$$\mathrm{Ker}(T) = \left\{\begin{bmatrix} x \\ y \\ -x-y \end{bmatrix}\right\} = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right\}$$

$$\mathrm{im}(S) = \left\{\begin{bmatrix} 0 \\ x \\ y \end{bmatrix}\right\} = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right\}$$

$$\mathrm{Ker}(S) = \left\{\begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}\right\} = \mathrm{Span}\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$

$$(S+T)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = S\begin{bmatrix} x \\ y \\ z \end{bmatrix} + T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y+z \\ x \\ y \end{bmatrix}$$

$$\mathrm{im}(S+T) = \mathbb{R}^3$$

$$\mathrm{Ker}(S+T) = \{\underline{0}\}$$

$$(S \circ T)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = S\left(T\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = S\begin{bmatrix} x+y+z \\ 0 \\ 0 \end{bmatrix}$$

$$\mathrm{im}(S \circ T) = \mathrm{Span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\} = \begin{bmatrix} 0 \\ x+y+z \\ 0 \end{bmatrix} \quad , \quad \mathrm{Ker}(S \circ T) = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right\}$$

$$(T \circ S)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = T\begin{bmatrix} 0 \\ x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ 0 \\ 0 \end{bmatrix}$$

$$\operatorname{Ker}(T \circ S) = \operatorname{Span}\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right\}$$

$$\operatorname{im}(T \circ S) = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$$

H.W Computations

Composition of rotations are rotations

$$R_\theta \cdot R_\theta = R_{(\theta+\theta)}$$

$$\left(R_\theta \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}\right)$$

$$R_\theta\left(R_\theta \begin{bmatrix} x \\ y \end{bmatrix}\right) = R_{(\theta+\theta)}\begin{bmatrix} x \\ y \end{bmatrix}$$

Notation: If $T \circ T = T$ then T is a Projection.

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2x - y - z \\ -x + 2y - z \\ -x - y + 2z \end{bmatrix}$$

Projection onto plane

$x + y + z = 0$

$$\operatorname{im}(T) = \operatorname{Span}\left\{\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}\right\}$$

$$\left(\text{Note: } \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} = -\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}\right)$$

$$\operatorname{Ker}(T) = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$$

Note: $\operatorname{Ker}(T) \cap \operatorname{im}(T) = \{\underline{0}\}$ so $\operatorname{Ker}(T \circ T) = \operatorname{Ker}(T)$

$$\mathbb{R}^3 \xrightarrow{T} \mathbb{R}^3 \xrightarrow{T} \mathbb{R}^3$$

$$(T\circ T)\begin{bmatrix} x\\ y\\ z\end{bmatrix} = T\left(\frac{1}{3}\begin{bmatrix} 2x-y-z\\ -x+2y-z\\ -x-y+2z\end{bmatrix}\right) = \frac{1}{9}\begin{bmatrix} 2(2x-y-z)-(-x+2y-z)-(-x-y-2z) \\ \sim \quad \sim \quad \sim \\ \sim \quad \sim \quad \sim \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 2x-y-z\\ -x+2y-z\\ -x-y+2z\end{bmatrix} = T\begin{bmatrix} x\\ y\\ z\end{bmatrix}$$

Exercise 13:

$S: P_n(\mathbb{R}) \longrightarrow P_n(\mathbb{R})$
by $S(p(x)) = p(x+1)$ Is S a linear Transformation?

<u>check</u> $S(0)=0$ ($\underline{0}=0$ function)

- $S(p(x)+q(x)) = S((p+q)(x))$
$= (p+q)(x+1)$
$= p(x+1) + q(x+1)$
$= S(p(x)) + S(q(x))$

- $S(k\cdot p(x)) = S((k\cdot p)(x))$
$= (k\cdot p)(x+1)$
$= k\cdot p(x+1)$
$= k\,S(p(x))$

. Is S an isomorphism?

→ Computing $\operatorname{im}(S)$, $\operatorname{Ker}(S)$ is tricky.
→ Look at S (function basis) — do we get a basis?

$$\left.\begin{array}{l} 1 \longrightarrow 1 \\ x \longrightarrow x+1 \\ x^2 \longrightarrow x^2+2x+1 \\ \vdots \\ x^n \longrightarrow x^n + n x^{n-1} + \cdots + 1 \end{array}\right\}$$ basis b/c Linearly indep.

→ $S^{-1}(p(x)) = p(x-1)$

<u>check</u>: $(S^{-1}\circ S)\,p(x) = S^{-1}(p(x+1))$
$= p(x-1+1)$
$= p(x).$

Standard Linear Transf.

$P(\mathbb{R}) \longrightarrow P(\mathbb{R})$ $\quad p \longrightarrow \frac{d}{dx} p$

$p \longrightarrow \int p\,dx$ (with $c=0$)

$p \longrightarrow p(k(x))$

$p \longrightarrow k(x)\cdot p(x)$

($k(x)$ Same fixed Polynom)

Chapter 10: Linear Transformation and Matrices.

10.0

Recall: Every linear Transf. $\mathbb{R} \xrightarrow{T} \mathbb{R}$

$$\text{is } T(x) = a_T \cdot x$$

$$\left(T = a_T^* \in \mathbb{R}^* \right)$$

. Every linear Transf. $\mathbb{R}^2 \longrightarrow \mathbb{R}$

$$\text{is } T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = a_T \cdot x + b_T \cdot y$$

$$\left(T = \begin{bmatrix} a \\ b \end{bmatrix}^* \in (\mathbb{R}^2)^* \right)$$

Next: $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} T_1\begin{bmatrix} x \\ y \end{bmatrix} \\ T_2\begin{bmatrix} x \\ y \end{bmatrix} \end{bmatrix} = \begin{bmatrix} a \cdot x + b \cdot y \\ c \cdot x + d \cdot y \end{bmatrix}$$

$$[T] = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\text{Matrix}}$$

$$T(\underline{v}) = \begin{bmatrix} - \Gamma_1 - \\ - \Gamma_2 - \end{bmatrix} \underline{v} = \begin{bmatrix} \Gamma_1 \cdot \underline{v} \\ \Gamma_2 \cdot \underline{v} \end{bmatrix}$$

→ $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ and Matrices

What does T look like?

→ Linear Transf. of
$$\begin{cases} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \longrightarrow T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} \\ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \longrightarrow T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \longrightarrow T\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} \end{cases}$$

$$\rightarrow T = \begin{bmatrix}x\\y\\z\end{bmatrix} = T\left(x\begin{bmatrix}1\\0\\0\end{bmatrix} + y\begin{bmatrix}0\\1\\0\end{bmatrix} + z\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$$

$$= x.T\begin{bmatrix}1\\0\\0\end{bmatrix} + y.T\begin{bmatrix}0\\1\\0\end{bmatrix} + z.T\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$= \begin{bmatrix}xa_{11} + ya_{12} + za_{13}\\ xa_{21} + ya_{22} + za_{23}\\ xa_{31} + ya_{32} + za_{33}\end{bmatrix}$$ * nice notation.

$T\begin{bmatrix}1\\0\\0\end{bmatrix}$ $T\begin{bmatrix}0\\1\\0\end{bmatrix}$ $T\begin{bmatrix}0\\0\\1\end{bmatrix}$ $\longrightarrow$ Column of T are T(basis vectors)

$$T \longleftrightarrow [T] = \underbrace{\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}}_{3\times 3 \text{ array of numbers}} \leftarrow \text{Matrix of } T$$

$a_{ij}\begin{cases} i = \text{row}\\ j = \text{Column}\end{cases}$

$$T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) \longleftrightarrow [T]\cdot\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}xa_{11} + ya_{12} + za_{13}\\ \vdots \\ \end{bmatrix}$$

Fancier Notation : $\underline{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\underline{e}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $\underline{e}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$

$$T\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}a_{11}\\a_{21}\\a_{31}\end{bmatrix} = a_{11}\begin{bmatrix}1\\0\\0\end{bmatrix} + a_{21}\begin{bmatrix}0\\1\\0\end{bmatrix} + a_{31}\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$T\underline{e}_1 = a_{11}\underline{e}_1 + a_{21}\underline{e}_2 + a_{31}\underline{e}_3$$

$$\Rightarrow T\underline{e}_j = \sum_{i=1} a_{ij}\,\underline{e}_i$$

if $\underline{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = x_1\underline{e}_1 + x_2\underline{e}_2 + x_3\underline{e}_3$

Then $T\underline{x} = x_1 T\underline{e}_1 + x_2 T\underline{e}_2 + x_3 T\underline{e}_3$

$$= \sum_{j=1} x_j\,(T\underline{e}_j) = \sum_{j=1}\sum_{i=1} a_{ij}\,x_j\,\underline{e}_i$$

Ex. $T\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}x+2z\\y-z\\z+y\end{bmatrix} \longrightarrow \begin{bmatrix}1\cdot x+0\cdot y+2\cdot z\\0\cdot x+1\cdot y-1\cdot z\\0\cdot x+1\cdot y+1\cdot z\end{bmatrix}$

$$T\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1+0\\0-0\\0+0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$T\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0+0\\1-0\\0+1\end{bmatrix} = \begin{bmatrix}0\\1\\1\end{bmatrix}$$

$$T\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0+2\\0-1\\1+0\end{bmatrix} = \begin{bmatrix}2\\-1\\1\end{bmatrix}$$

$$\Rightarrow [T] = \begin{bmatrix}1&0&2\\0&1&-1\\0&1&1\end{bmatrix}$$

Recall Dot Product $\begin{bmatrix}a\\b\\c\end{bmatrix}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix} = ax+by+cz$

$$= \begin{bmatrix}a\\b\\c\end{bmatrix}^{*}\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)$$

Computational Note

$$[T]\cdot v = T(v) = \begin{bmatrix}—r_1—\\—r_2—\\—r_3—\end{bmatrix}\cdot \underline{v} = \begin{bmatrix}\underline{r_1}\cdot\underline{v}\\\underline{r_2}\cdot\underline{v}\\\underline{r_3}\cdot\underline{v}\end{bmatrix}$$

Think of $[T]$ as collection of rows

Ex. T from previous example.

$T\cdot\begin{bmatrix}1\\2\\3\end{bmatrix} = ?$

$$\Rightarrow \begin{bmatrix}1&0&2\\0&1&-1\\0&1&1\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}1\cdot1+0\cdot2+2\cdot3\\0\cdot1+1\cdot2+(-1)\cdot3\\0\cdot1+1\cdot2+1\cdot3\end{bmatrix} = \begin{bmatrix}7\\-1\\5\end{bmatrix}$$

Ex. $S\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\x\\y\end{bmatrix}$

$$[S] = \begin{bmatrix}0&0&0\\1&0&0\\0&1&0\end{bmatrix} \begin{matrix}\longleftarrow x,y,z \text{ Coeff. of}\\ \longleftarrow x,y,z \quad \sim \quad \sim\\ \longleftarrow x,y,z \quad \sim \quad \sim\end{matrix} \begin{bmatrix}0\\x\\y\end{bmatrix}$$

Ex. $S\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}0\\1\\2\end{bmatrix}$ ✓

Goal Do all Linear Transf. stuff with matrices.

$T(\underline{v}) = [T]\cdot\underline{v}$ ✓

I. $S+T \longleftrightarrow [S+T] = ??$

II. $k\cdot T \longleftrightarrow [kT] = ??$

III. $S\circ T \longleftrightarrow [S\circ T] = ??$

I. $$[S+T]\begin{bmatrix}x\\y\\z\end{bmatrix} = (S+T)\begin{bmatrix}x\\y\\z\end{bmatrix} = S\begin{bmatrix}x\\y\\z\end{bmatrix} + T\begin{bmatrix}x\\y\\z\end{bmatrix} = [S]\cdot\begin{bmatrix}x\\y\\z\end{bmatrix} + [T]\cdot\begin{bmatrix}x\\y\\z\end{bmatrix}$$
$$= [S] + [T]$$

Redux. $$[S+T] = \begin{bmatrix} | & | & | \\ (S+T)\underline{e}_1 & (S+T)\underline{e}_2 & (S+T)\underline{e}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ (S\underline{e}_1+T\underline{e}_1) & (S\underline{e}_2+T\underline{e}_2) & (S\underline{e}_3+T\underline{e}_3) \\ | & | & | \end{bmatrix}$$

= add corresponding entries.

II. $$[kT] = \begin{bmatrix} | & | & | \\ (kT)e_1 & (kT)e_2 & (kT)e_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ k\cdot(Te_1) & k\cdot(Te_2) & k\cdot(Te_3) \\ | & | & | \end{bmatrix}$$

= multiply <u>all</u> entries by k.

III. $[S\circ T]\,\underline{x} = [S]\cdot[T]\,\underline{x}$ (define $[S]\cdot[T]$ so that this happens.)

$$[S\circ T] = \begin{bmatrix} | & | & | \\ (S\circ T)\underline{e}_1 & (S\circ T)e_2 & (S\circ T)e_3 \\ | & | & | \end{bmatrix}$$
$$= \begin{bmatrix} | & | & | \\ S(T\underline{e}_1) & S(T\underline{e}_2) & S(T\underline{e}_3) \\ | & | & | \end{bmatrix}$$
$$= \begin{bmatrix} | & | & | \\ [S]T\underline{e}_1 & [S]T\underline{e}_2 & [S]T\underline{e}_3 \\ | & | & | \end{bmatrix}$$

Rows of [S]

Column of T

$$S = \begin{bmatrix} - & s_1 & - \\ - & s_2 & - \\ - & s_3 & - \end{bmatrix}, \quad T = \begin{bmatrix} | & | & | \\ t_1 & t_2 & t_3 \\ | & | & | \end{bmatrix}$$

$$= \begin{bmatrix} | & | & | \\ [S]t_1 & [S]t_2 & [S]t_3 \\ | & | & | \end{bmatrix}$$

dot product.

$$= \begin{bmatrix} s_1 \cdot t_1 & s_1 \cdot t_2 & s_1 \cdot t_3 \\ s_2 \cdot t_1 & s_2 \cdot t_2 & s_2 \cdot t_3 \\ s_3 \cdot t_1 & s_2 \cdot t_2 & s_3 \cdot t_3 \end{bmatrix}$$

Result. Matrix Product is defined so that:

if
$$\begin{cases} [S] = \begin{bmatrix} a_{11} & \cdots \\ \vdots & \end{bmatrix} = (a_{ij}) \\ [T] = \begin{bmatrix} b_{11} & \cdots \\ \vdots & \end{bmatrix} = (b_{ij}) \end{cases}$$

Then:
$$[S][T] = \begin{bmatrix} \sum_{k=1} a_{1k} b_{k1} & \cdots \\ \vdots & \end{bmatrix} = \left(\sum_{k=1} a_{ik} b_{kj} \right)$$

Notations

- i will always be row index
- j will be always col. index
- k will be used for mixed index

Ex. $S\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ x \\ y \end{bmatrix} \rightsquigarrow [S] = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+2z \\ -x+y \\ y+z \end{bmatrix} \rightsquigarrow [T] = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$[S \cdot T] = \begin{bmatrix} 0 \\ x+2y \\ -x+y \end{bmatrix}$

$$[S][T] = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 2 \\ -1 & 1 & 0 \end{bmatrix}$$

✓

$[T \circ S] = T\begin{bmatrix} 2y \\ x \\ x+y \end{bmatrix} \Rightarrow [T][S] = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

Matrix mult. is not commutative !!!

Ex. $S\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\x\\y\end{bmatrix}$ $\qquad [S]=\begin{bmatrix}0&0&0\\1&0&0\\0&1&0\end{bmatrix}$

$S.S\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\x\end{bmatrix}$ $\qquad [S][S]=\begin{bmatrix}0&0&0\\0&0&0\\1&0&0\end{bmatrix}$

$S.S.S\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ $\qquad [S][S][S]=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$

Ex. $T\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}x+2y+3z\\4x+5y+6z\\7x+8y+9z\end{bmatrix} \longrightarrow [T]=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}$

$P\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}z\\x\\2y\end{bmatrix} \longrightarrow [P]=\begin{bmatrix}0&0&1\\1&0&0\\0&2&0\end{bmatrix}$

$$[P][T]=\begin{bmatrix}0&0&1\\1&0&0\\0&2&0\end{bmatrix}\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}=\begin{array}{c}\begin{bmatrix}7&8&9\\1&2&3\\8&10&12\end{bmatrix}\end{array}\begin{array}{l}\leftarrow \text{Row 3 of } [T]\\ \leftarrow \text{Row 1 of } [T]\\ \leftarrow 2\,\text{Row 2 of } [T]\end{array}$$

$P\begin{pmatrix}-t_1-\\-t_2-\\-t_3-\end{pmatrix}=\begin{pmatrix}-t_3-\\-t_1-\\-2t_2-\end{pmatrix}$ $\qquad$ P.A does row operations on A

$$[T][P]=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\begin{bmatrix}0&0&1\\1&0&0\\0&2&0\end{bmatrix}=\begin{bmatrix}2&6&1\\5&12&4\\8&18&7\end{bmatrix}$$

col 1 of T

col 2 of T $\qquad$ 2 col 3 of T

$\begin{pmatrix}|&|&|\\t_1&t_2&t_3\\|&|&|\end{pmatrix}P=\begin{pmatrix}|&|&|\\t_2&2t_3&t_1\\|&|&|\end{pmatrix}$ $\qquad$ A.P does col operations on A.

10.3 . Transformations $T: \mathbb{R}^m \to \mathbb{R}^n$

Same theory : $T \longleftarrow [T] = \underbrace{\begin{bmatrix} | & | & & | \\ Te_1 & Te_2 & \cdots & Te_m \\ | & | & & | \end{bmatrix}}_{m} \Big\} n$

Note on dimensions

$$\underbrace{\underset{(n \times m)}{[T]} \cdot \underset{m}{\underline{v}}}_{n}$$

Scaling $k \cdot [T]$

Sums $[T] + [S]$ } All the Same

Products $[T] \cdot [S]$

Note Can only add if sizes match

$$\underset{m \times n}{[T]} + \underset{m \times n}{[S]} = \underset{m \times n}{[T+S]}$$

$$\mathbb{R}^n \to \mathbb{R}^n \quad \mathbb{R}^n \to \mathbb{R}^n \quad \mathbb{R}^n \to \mathbb{R}^n$$

In particular, cannot do

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 7 & 0 \\ -1 & 2 & 0 \end{bmatrix} \quad X$$

Note. Can only multiply if "input" and "output" sizes match.

$$[T] \cdot [S] = [T \circ S]$$

$$\mathbb{R}^k \to \mathbb{R}^n \quad \mathbb{R}^n \to \mathbb{R}^m \quad \mathbb{R}^n \to \mathbb{R}^m$$

$$m \times k \quad k \times n \quad m \times n$$

In particular, no : $\underset{3 \times 2}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 1 & 2 \end{bmatrix}} \cdot \underset{3 \times 2}{\begin{bmatrix} 1 & 2 \\ 1 & 7 \\ 2 & 1 \end{bmatrix}}$

- Can only do A^2 if A is $n \times n$ ("square")

$\mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ is a V.S $\longrightarrow$ $Mat_{n\times m}$ is a V.S

$\left\{ \begin{matrix} \text{Linear transf.} \\ T: \mathbb{R}^m \to \mathbb{R}^n \end{matrix} \right\}$

$$Mat_{n\times m} = \left\{ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & & a_{nm} \end{bmatrix} \right\}$$

$$\text{Standard basis} = \left\{ E_{ij} = 0 \text{ at all positions except } a_{ij} = 1 \right\}$$

$E(i,j)$: "Elementary Matrices"

Prop: $\dim(Mat_{n\times m}) = n\cdot m = \dim(Mat_{m\times n})$

transpose

Cor: $\dim(\mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)) = n\cdot m = \dim(\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m))$

Note. $\mathcal{L}(\mathbb{R}, \mathbb{R}^n) = \mathbb{R}^n \qquad \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right\}$

$\updownarrow$

$\mathcal{L}(\mathbb{R}^n, \mathbb{R}) = (\mathbb{R}^n)^* \qquad \left\{ \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \right\}$

Ex. $\underset{1\times 3}{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}} \cdot \underset{3\times 1}{\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}} = 1\cdot 4 + 2\cdot 5 + 3\cdot 6$

Ex. $\underset{3\times 1}{\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}} \underset{1\times 3}{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}} = \underset{3\times 3}{\begin{bmatrix} 4\cdot 1 & 4\cdot 2 & 4\cdot 3 \\ 5\cdot 1 & 5\cdot 2 & 5\cdot 3 \\ 6\cdot 1 & 6\cdot 2 & 6\cdot 3 \end{bmatrix}}$